Bessel Equation of Order One-Half. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting \( \nu = \frac{1}{2} \) in Eq. (1) gives

\[
L[y] = x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0. \tag{16}
\]

If we substitute the series (3) for \( y = \phi(r, x) \), we obtain

\[
L[\phi](r, x) = \sum_{n=0}^{\infty} \left[ (r + n)(r + n - 1) + (r + n) - \frac{1}{8} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2}
\]

\[
= (r^2 - \frac{1}{4}) a_0 x^r + \left[ (r + 1)^2 - \frac{1}{4} \right] a_1 x^{r+1}
\]

\[
+ \sum_{n=2}^{\infty} \left\{ \left[ (r + n)^2 - \frac{1}{4} \right] a_n + a_{n-2} \right\} x^{r+n} = 0. \tag{17}
\]

The roots of the indicial equation are \( r_1 = \frac{1}{2} \), \( r_2 = -\frac{1}{2} \), hence the roots differ by an integer. The recurrence relation is

\[
\left[ (r + n)^2 - \frac{1}{4} \right] a_n = -a_{n-2}, \quad n \geq 2. \tag{18}
\]

Corresponding to the larger root \( r_1 = \frac{1}{2} \) we find from the coefficient of \( x^{r+1} \) in Eq. (17) that \( a_1 = 0 \). Hence, from Eq. (18), \( a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0 \). Further, for \( r = \frac{1}{2} \),

\[
a_n = -\frac{a_{n-2}}{n(n+1)}, \quad n = 2, 4, 6, \ldots,
\]

or letting \( n = 2m \), we obtain

\[
a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)}, \quad m = 1, 2, 3, \ldots.
\]

By solving this recurrence relation we find that

\[
a_2 = \frac{-a_0}{3!}, \quad a_4 = \frac{a_0}{5!}, \ldots
\]

and, in general,

\[
a_{2m} = (-1)^m \frac{a_0}{(2m+1)!}, \quad m = 1, 2, 3, \ldots.
\]

Hence, taking \( a_0 = 1 \), we obtain

\[
y_1(x) = x^{1/2} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!} \right] = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}, \quad x > 0. \tag{19}
\]

The power series in Eq. (19) is precisely the Taylor series for \( \sin x \); hence one solution of the Bessel equation of order one-half is \( x^{-1/2} \sin x \). The Bessel function of the first kind of order one-half, \( J_{1/2} \), is defined as \( (2/\pi)^{1/2} y_1 \). Thus

\[
J_{1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \sin x, \quad x > 0. \tag{20}
\]