\[ a(\mu, f) = c'_1 G(\chi_\sigma, t) t^{-s_2} \]

\[ \cdot \sum_{|\nu| m} \chi_\sigma(\nu)^{m \nu^2 - 1} a_f \left( \frac{n^2}{\nu^2} \right) |Q(\xi, \xi)|, \]

with \( \chi_\sigma \) the Dirichlet character given by

\[ \chi_\sigma(x) = \sigma(x)(-1/x)^{2s} \lambda_{\overline{Q}, \sigma}(x) \]

and \( G(\chi_\sigma, t) \), the Gauss sum given by

\[ G(\chi_\sigma, t) = \sum_{\nu \mod t} \chi_\sigma(\nu) e^{2\pi i \nu / t} \]

(here \( c'_1 \) is a nonzero constant depending only on \( s \)).

Then following the well-known methods in automorphic form theory, it is possible to associate a Dirichlet series to the automorphic function \( F_\ell \) (\( |\varphi, L, v, \chi_\sigma, \eta(t)\)). The cases \( k = 3, 4 \) have been studied extensively in [10] and [6], so in the ensuing discussion we assume that \( k \geq 5 \). In particular we let

\[ R(\tilde{s}, f) = \sum_{Q \in (|F_\ell| \epsilon(Q) \epsilon_+)} a(Q, f) \left( \frac{1}{\epsilon(Q)} \right) |Q(\Omega, \Omega)|^{-s}, \]

where \( \{(|F_\ell| \epsilon(Q) \epsilon_+) \} \) denotes the set of equivalence classes of \( O(Q, F_1 + F_1)^4 \cap O(Q)_L \) in \( (|F_\ell| \epsilon(Q) \epsilon_+ \) and \( \epsilon(Q) \) is the order of the finite subgroup of \( O(Q, F_1 + F_1)^4 \cap O(Q)_L \) which fixes \( Q \).

Then from (6–10) we deduce immediately

**Proposition 6.2 [16]** (With the same hypotheses as in Theorem 6.1 and \( k \geq 5 \). Let \( \tilde{s} \in \mathbb{C} \) so that \( \text{Re}(\tilde{s}) \) is sufficiently large. Then we have the identity:

\[ R(\tilde{s}, f) = d_1 G(\chi_\sigma, t) t^{2s - s_2} 2^s L(\chi_\sigma, 2\tilde{s} + 1 - s_2) \]

\[ \sum_{n \in \mathbb{Z} \mid \mathbb{Z} \setminus 1} a_f(-n) M(Q_1, \mathcal{L}, n) |n|^{-s}, \]

where \( M(Q_1, \mathcal{L}, n) \) is the Siegel mass number of the form \( Q_1 \) (\( = Q \) restricted to \( F_1 + F_1 \)) relative to the lattice \( \mathcal{L} \) on the quadric of level \( n \) (i.e. \( M(Q_1, \mathcal{L}, n) = \sum \epsilon_\xi \xi \epsilon_{\xi}^{-1} \), where \( \xi, \ldots, \xi_{\mathbb{N}} \) run through a set of representatives of \( O(Q, F_1 + F_1)^4 \cap O(Q)_L \) orbits in \( \mathcal{L} \) \( X \in \mathbb{R}^{k-2} [Q(X, X) = n] \)) and \( d_1 \) a nonzero constant dependent only on \( s_2 \) (recall here that \( s_2 = s + \frac{1}{2} k - 1 \)). Also \( L(\tilde{s}, \tilde{s}) \) is the classical \( L \) function associated to the Dirichlet character \( \tilde{s} \).

Thus we have expressed \( R(\tilde{s}, f) \) as the product of elementary functions (i.e. \( a_f \)), an \( L \) function, and the Rankin convolution of 2 Dirichlet series (i.e. the Dirichlet series \( D(\tilde{s}, f) = \sum_{n \geq 1} a_f(n) n^{-s} \) and Siegel's zeta function \( \zeta_{-}(Q_1, \mathcal{L}, \tilde{s}) = \sum_{n \in \mathbb{Z} \mid \mathbb{Z} \setminus 1} M(Q_1, \mathcal{L}, n) |n|^{-s}. \)

The analytic nature of the function \( R(\tilde{s}, f) \) can then be determined easily from Proposition 6.1. If we let \( R^*(\tilde{s}, f) = \{ \pi^{-\frac{s_2}{2}} (\tilde{s} - \frac{k}{2} + 2) \} \), then \( R^*(\tilde{s}, f) \) can be analytically continued to the whole \( \tilde{s} \) plane (\( \Gamma \), the gamma function).

**Remark 6.2.** Using (6–12), it is possible to deduce a type of Euler product expansion of \( R(\tilde{s}, f) \). It suffices to study the possible Euler product properties of the Rankin convolution of \( D(\tilde{s}, f) \) and \( \zeta_{-}(Q_1, \mathcal{L}, \tilde{s}) \). However if \( k \) is even and both \( D(\tilde{s}, f) \) and \( \zeta_{-}(Q_1, \mathcal{L}, \tilde{s}) \) admit the usual Euler product of the \( GL_2 \) theory, then the Rankin
convolution of these series can be expressed as an Euler product with numerator of degree 2 and denominator of degree 4 for almost all primes \( p \). (For suitable choice of \( Q_1 \) and \( \mathfrak{L} \), \( \zeta_-(Q_1, \mathfrak{L}, \mathcal{S}) \) is a finite sum of Euler products of the \( \text{GL}_2 \) theory, see [7].) On the other hand, if \( k \) is odd then \( D(\mathcal{S}, f) \) and \( \zeta_-(Q_1, \mathfrak{L}, \mathcal{S}) \) do not have the usual \( \text{GL}_2 \) type Euler product. However in [18] a modified theory of Euler products is set forth for \( \text{SL}_2 \) automorphic forms of semi-integral weight. In particular if \( f \in S_{2k}(I_0(t), \beta) \) is a Hecke eigenfunction in the sense of [18], then the partial Dirichlet series \( \sum_{n \geq 1} a_n(\lfloor d \rfloor n^2) n^{-s} \) (\( d \), the discriminant of an imaginary quadratic extension of \( \mathcal{O} \)) can be expressed as an Euler product with numerator of degree 1 and denominator of degree 2 for almost all primes \( p \). And it is possible for suitable \( \mathfrak{L} \) to find a similar Euler product for \( \sum_{n \geq 1} M(Q_1, \mathfrak{L}, \lfloor d \rfloor n^2) M(Q_1, \mathfrak{L}, \lfloor d \rfloor n^2) n^{-s} \). Then by purely algebraic methods, one can show that the Rankin convolution \( \sum_{n \geq 1} a_n(\lfloor d \rfloor n^2) \cdot M(Q_1, \mathfrak{L}, \lfloor d \rfloor n^2) n^{-s} \) is an Euler product with numerator of degree 3 and denominator of degree 4 for almost all primes \( p \) (see [16] for the case \( k = 5 \) where \( O(3, 2) \) is locally isomorphic to \( \text{Sp}_2(\mathbb{R}) \)).

Notes. For a complete bibliography on “hyperboloid analysis”, we refer the reader to [21]. Also for an adelic treatment of the Weil representation and automorphic forms, see [4].

We use the terminology that a function \( \phi \) vanishes at a cusp \( \gamma(\infty) = a \) if \( (\phi \mid \gamma)(z) = (cz + d)^{-1} \phi(\gamma(z)) \) has an expansion of the form

\[
\sum_{n \geq 0} c_n e^{2\pi \sqrt{-1}(a + c) N z}
\]

with \( c_0 = 0 \) when \( \kappa = 0 \)

(here \( \kappa \) is the ramification of the multiplier at \( a \) and \( N \) is the smallest positive integer so that

\[
\gamma \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix} \gamma^{-1} \in \Gamma_1
\]

\( \Gamma_1 \) the arithmetic group in question) (see [18]).

BIBLIOGRAPHY