Corresponding to the root \( r_2 = -\frac{1}{2} \) it is possible that we may have difficulty in computing \( a_1 \) since \( N = r_1 - r_2 = 1 \). However, from Eq. (17) for \( r = -\frac{1}{2} \) the coefficients of \( x^n \) and \( x^{n+1} \) are both zero regardless of the choice of \( a_0 \) and \( a_1 \). Hence \( a_0 \) and \( a_1 \) can be chosen arbitrarily. From the recurrence relation (18) we obtain a set of even-numbered coefficients corresponding to \( a_0 \) and a set of odd-numbered coefficients corresponding to \( a_1 \). Thus no logarithmic term is needed to obtain a second solution in this case. It is left as an exercise to show that, for \( r = -\frac{1}{2} \),

\[
\begin{align*}
\frac{a_{2n}}{(2n)!} &= \frac{(-1)^n a_0}{(2n)!}, \\
\frac{a_{2n+1}}{(2n+1)!} &= \frac{(-1)^n a_1}{(2n+1)!},
\end{align*}
\]

\( n = 1, 2, \ldots \)

Hence

\[
y_2(x) = x^{-1/2} \left[ a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right]
\]

\[
= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \quad x > 0.
\]

(21)

The constant \( a_1 \) simply introduces a multiple of \( y_1(x) \). The second linearly independent solution of the Bessel equation of order one-half is usually taken to be the solution for which \( a_0 = (2/\pi)^{1/2} \) and \( a_1 = 0 \). It is denoted by \( J_{-1/2} \). Then

\[
J_{-1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos x, \quad x > 0.
\]

(22)

The general solution of Eq. (16) is \( y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) \).

By comparing Eqs. (20) and (22) with Eqs. (14) and (15) we see that, except for a phase shift of \( \pi/4 \), the functions \( J_{-1/2} \) and \( J_{1/2} \) resemble \( J_0 \) and \( Y_0 \), respectively, for large \( x \). The graphs of \( J_{1/2} \) and \( J_{-1/2} \) are shown in Figure 5.8.4.

![Figure 5.8.4](image-url) The Bessel functions \( J_{1/2} \) and \( J_{-1/2} \).