in four steps:

\[
\begin{align*}
gcd(m, n) &= gcd(n_0, n_1) \\
gecd(n_0, n_1) &= gcd(n_1, n_2) \\
gecd(n_1, n_2) &= gcd(n_2, n_3) \\
gecd(n_2, n_3) &= gcd(n_3, n_4) \\
gecd(n_3, n_4) &= gcd(n_4, 0) = n_4
\end{align*}
\]

Then we have

\[
\begin{align*}
n_4 &= n_4 &\equiv K(n_4) \\
n_3 &= q_4n_4 &\equiv K(q_4)n_4; \\
n_2 &= q_3n_3 + n_4 = K(q_3, q_4)n_4; \\
n_1 &= q_2n_2 + n_3 = K(q_2, q_3, q_4)n_4; \\
n_0 &= q_1n_1 + n_2 = K(q_1, q_2, q_3, q_4)n_4
\end{align*}
\]

In general, if Euclid’s algorithm finds the greatest common divisor \(d\) in \(k\) steps, after computing the sequence of quotients \(q_1, \ldots, q_k\), then the starting numbers were \(K(q_1, q_2, \ldots, q_k)d\) and \(K(q_2, \ldots, q_k)d\). (This fact was noticed early in the eighteenth century by Thomas Fantet de Lagny [190], who seems to have been the first person to consider continuants explicitly. Lagny pointed out that consecutive Fibonacci numbers, which occur as continuants when the \(q\)’s take their minimum values, are therefore the smallest inputs that cause Euclid’s algorithm to take a given number of steps.)

Continuants are also intimately connected with continued fractions, from which they get their name. We have, for example,

\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}} = \frac{K(a_0, a_1, a_2, a_3)}{K(a_1, a_2, a_3)}.
\]

(6.135)

The same pattern holds for continued fractions of any depth. It is easily proved by induction; we have, for example,

\[
\frac{K(a_0, a_1, a_2, a_3 + 1/a_4)}{K(a_1, a_2, a_3 + 1/a_4)} = \frac{K(a_0, a_1, a_2, a_3, a_4)}{K(a_1, a_2, a_3, a_4)},
\]

because of the identity

\[
K_n(x_1, \ldots, x_{n-1}, x_n + y) = K_n(x_1, \ldots, x_{n-1}, x_n) + K_{n-1}(x_1, \ldots, x_{n-1})y
\]

(6.136)

(This identity is proved and generalized in exercise 30.)