Bessel Equation of Order One. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting \( \nu = 1 \) in Eq. (1) gives
\[
L[y] = x^2 y'' + xy' + (x^2 - 1)y = 0. \tag{23}
\]
If we substitute the series (3) for \( y = \phi(r, x) \) and collect terms as in the preceding cases, we obtain
\[
L[\phi](r, x) = a_0 (r^2 - 1)x' + a_1 [(r + 1)^2 - 1]x^{r+1} \\
+ \sum_{n=2}^{\infty} [(r + n)^2 - 1]a_n + a_{n-2}]x^{r+n} = 0. \tag{24}
\]
The roots of the indicial equation are \( r_1 = 1 \) and \( r_2 = -1 \). The recurrence relation is
\[
(r + n)^2 - 1]a_n (r) = -a_{n-1}(r), \quad n \geq 2. \tag{25}
\]
Corresponding to the larger root \( r = 1 \) the recurrence relation becomes
\[
a_n = -\frac{a_{n-2}}{(n + 2)n}, \quad n = 2, 3, 4, \ldots.
\]
We also find from the coefficient of \( x^{r+1} \) in Eq. (24) that \( a_1 = 0 \); hence from the recurrence relation \( a_3 = a_5 = \cdots = 0 \). For even values of \( n \), let \( n = 2m \); then
\[
a_{2m} = -\frac{a_{2m-2}}{(2m + 2)(2m)} = -\frac{a_{2m-2}}{2^2(m + 1)m}, \quad m = 1, 2, 3, \ldots.
\]
By solving this recurrence relation we obtain
\[
a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m + 1)!}, \quad m = 1, 2, 3, \ldots. \tag{26}
\]
The Bessel function of the first kind of order one, denoted by \( J_1 \), is obtained by choosing \( a_0 = 1/2 \). Hence
\[
J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m + 1)! m!}. \tag{27}
\]
The series converges absolutely for all \( x \), so the function \( J_1 \) is analytic everywhere.

In determining a second solution of Bessel’s equation of order one, we illustrate the method of direct substitution. The calculation of the general term in Eq. (28) below is rather complicated, but the first few coefficients can be found fairly easily. According to Theorem 5.7.1 we assume that
\[
y_2(x) = a J_1(x) \ln x + x^{-1} \left[ 1 + \sum_{n=1}^{\infty} c_n x^n \right], \quad x > 0. \tag{28}
\]
Computing \( y'_2(x), y''_2(x) \), substituting in Eq. (23), and making use of the fact that \( J_1 \) is a solution of Eq. (23) give
\[
2ax J'_1(x) + \sum_{n=0}^{\infty} [(n - 1)(n - 2)c_n + (n - 1)c_n - c_n]x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0. \tag{29}
\]