9. Consider the Bessel equation of order \( \nu \),

\[
x^2 y'' + xy' + (x^2 - \nu^2) y = 0,
\]

\( x > 0 \).

Take \( \nu \) real and greater than zero.
(a) Show that \( x = 0 \) is a regular singular point, and that the roots of the indicial equation are \( \nu \) and \(-\nu\).
(b) Corresponding to the larger root \( \nu \), show that one solution is

\[
y_1(x) = x^\nu \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{m!(1+\nu)(2+\nu) \cdots (m-1+\nu)(m+\nu)} \right].
\]

(c) If \( 2\nu \) is not an integer, show that a second solution is

\[
y_2(x) = x^{-\nu} \left[ 1 + \sum_{m=1}^{\infty} \frac{\nu^m x^{2m}}{m!(1-\nu)(2-\nu) \cdots (m-1-\nu)(m-\nu)} \right].
\]

Note that \( y_1(x) \to 0 \) as \( x \to 0 \), and that \( y_2(x) \) is unbounded as \( x \to 0 \).
(d) Verify by direct methods that the power series in the expressions for \( y_1(x) \) and \( y_2(x) \) converge absolutely for all \( x \). Also verify that \( y_2 \) is a solution provided only that \( \nu \) is not an integer.

10. In this section we showed that one solution of Bessel’s equation of order zero,

\[
L[y] = x^2 y'' + xy' + x^2 y = 0,
\]

is \( J_0 \), where \( J_0(x) \) is given by Eq. (7) with \( a_0 = 1 \). According to Theorem 5.7.1 a second solution has the form \((x > 0)\)

\[
y_2(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} b_n x^n.
\]

(a) Show that

\[
L[y_2](x) = \sum_{n=2}^{\infty} n(n - 1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2x J_0'(x).
\]

(b) Substituting the series representation for \( J_0(x) \) in Eq. (i), show that

\[
b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty} (n^2 b_n + b_{n-2}) x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}. \tag{ii}
\]

(c) Note that only even powers of \( x \) appear on the right side of Eq. (ii). Show that \( b_1 = b_3 = \cdots = 0 \), \( b_2 = 1/2^2(1!)^2 \), and that

\[
(2n)^2 b_{2n} + b_{2n-2} = -2(-1)^n(2n)/2^{2n}(n!)^2, \quad n = 2, 3, 4, \ldots
\]

Deduce that

\[
b_4 = \left(-\frac{1}{2} + \frac{1}{2} \right), \quad \text{and} \quad b_6 = \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{3} \right).
\]

The general solution of the recurrence relation is \( b_{2n} = (-1)^{n+1} H_n/2^{2n}(n!)^2 \). Substituting for \( b_n \) in the expression for \( y_2(x) \) we obtain the solution given in Eq. (10).

11. Find a second solution of Bessel’s equation of order one by computing the \( c_n(r_2) \) and \( a \) of Eq. (24) of Section 5.7 according to the formulas (19) and (20) of that section. Some guidelines along the way of this calculation are the following. First, use Eq. (24) of this