section to show that \(a_2(-1)\) and \(a_2'(-1)\) are 0. Then show that \(c_n(-1) = 0\) for \(n = 3, 5, \ldots\). Finally, use Eq. (25) to show that
\[
a_{2m}(r) = \frac{(-1)^m a_0}{(r + 1)(r + 3)^2 \cdots (r + 2m - 1)^2 (r + 2m + 1)},
\]
for \(m = 1, 2, 3, \ldots\), and calculate
\[
c_{2m}(-1) = (-1)^m+1 \left( H_m + H_{m-1} \right)/2^{2m} m!(m - 1)!.
\]

12. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show that a solution of
\[
x^2 y'' + (\alpha^2 \beta^2 x^2 + 1 - \nu^2 \beta^2) y = 0, \quad x > 0
\]
is given by \(y = x^{1/2} f(ax^\beta)\) where \(f(\xi)\) is a solution of the Bessel equation of order \(\nu\).

13. Using the result of Problem 12 show that the general solution of the Airy equation
\[
y'' - xy = 0, \quad x > 0
\]
is \(y = x^{1/2} \left[ c_1 f_1(\frac{2}{3} ix^{3/2}) + c_2 f_2(\frac{2}{3} ix^{3/2}) \right]\) where \(f_1(\xi)\) and \(f_2(\xi)\) are linearly independent solutions of the Bessel equation of order one-third.

14. It can be shown that \(J_{\nu}\) has infinitely many zeros for \(x > 0\). In particular, the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 5.8.1). Let \(\lambda_j, j = 1, 2, 3, \ldots\), denote the zeros of \(J_{\nu}\); it follows that
\[
J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}
\]
Verify that \(y = J_0(\lambda_j x)\) satisfies the differential equation
\[
y'' + \frac{1}{x} y' + \lambda_j^2 y = 0, \quad x > 0.
\]
Hence show that
\[
\int_0^1 x J_0(\lambda_j x) J_0(\lambda_j x) \, dx = 0 \quad \text{if} \quad \lambda_j \neq \lambda_j.
\]
This important property of \(J_0(\lambda_j x)\), known as the orthogonality property, is useful in solving boundary value problems.

**Hint:** Write the differential equation for \(J_0(\lambda_j x)\). Multiply it by \(x J_0(\lambda_j x)\) and subtract it from \(x J_0(\lambda_j x)\) times the differential equation for \(J_0(\lambda_j x)\). Then integrate from 0 to 1.

**REFERENCES**


Proofs of Theorems 5.3.1 and 5.7.1 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, or Chapters 3 and 4 of:


Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 21 of Section 5.4. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of.