44. Prove the converse of Cassini’s identity (6.106): If \( k \) and \( m \) are integers such that \( m^2-km-k^2 = 1 \), then there is an integer \( n \) such that \( k = \pm F_n \) and \( m = \pm F_{n+1} \).

45. Use the repertoire method to solve the general recurrence

\[
X_0 = a; \quad X_1 = \beta; \quad X_n = X_{n-1} + X_{n-2} + \gamma n + \delta.
\]

46. What are \( \cos 36^\circ \) and \( \cos 72^\circ \)?

47. Show that

\[
2^{n-1}F_n = \sum_k \binom{n}{2k+1} 5^k,
\]

and use this identity to deduce the values of \( F_n \mod p \) and \( F_{p+1} \mod p \) when \( p \) is prime.

48. Prove that zero-valued parameters can be removed from continuant polynomials by collapsing their neighbors together:

\[
K_n(x_1, \ldots, x_{m-1}, 0, x_{m+1}, \ldots, x_n) = K_{n-2}(x_1, \ldots, x_{m-2}, x_{m-1}+x_{m+1}, x_{m+2}, \ldots, x_n), \quad 1 < m < n.
\]

49. Find the continued fraction representation of the number \( \sum_{n \geq 1} 2^{-\lfloor n\phi \rfloor} \).

50. Define \( f(n) \) for all positive integers \( n \) by the recurrence

\[
\begin{align*}
f(1) &= 1; \\
f(2n) &= f(n); \\
f(2n+1) &= f(n) + f(n+1).
\end{align*}
\]

a. For which \( n \) is \( f(n) \) even?

b. Show that \( f(n) \) can be expressed in terms of continuants.

**Exam problems**

51. Let \( p \) be a prime number.

a. Prove that \( \{p\} \equiv \{p\} \equiv 0 \pmod{p} \), for \( 1 < k < p \).

b. Prove that \( \{p^{-1}\} \equiv 1 \pmod{p} \), for \( 1 < k < p \).

c. Prove that \( \{-2/p\} \equiv 2^{-2} \pmod{p} \).

d. Prove that if \( p > 3 \) we have \( \{p/2\} \equiv 0 \pmod{p^2} \). Hint: Consider \( p^2 \).

52. Let \( H_n \) be written in lowest terms as \( a_n/b_n \).

a. Prove that \( p/b_n \iff p\mid a_{1/(n+p)} \), if \( p \) is prime.

b. Find all \( n > 0 \) such that \( a_n \) is divisible by 5.