SRG with loop outer regions - satisfy all the desirable conditions when the loop regions form a linearly independent set of size \#edges - \#nodes + 1 - i.e. a cycle basis. Our contribution can be seen as a refinement of the criteria proposed in [31]. In particular, we show that to satisfy the desirable conditions (e.g. non-singularity and sum of counting numbers equal 1), the set of loop regions must form a fundamental cycle basis. We also propose a condition called “tree-robustness” that further restricts the choice of loop regions. The idea of tree-robustness is to require GBP on a SRG to be exact on every possible tree embedded in the Markov Network, where an embedded tree is obtained by zeroing out the interactions on the off-tree edges. For loop SRGs, we show that this idea can be translated to an equivalent problem in graph theory, namely that of finding tree-robust cycle bases. We then characterize the class of tree robust cycle bases and identify two families of tree robust cycle bases. Finally, we demonstrate the performance of GBP on Loop-SRGs constructed to satisfy tree-robustness.

2 Generalized Belief Propagation and Structured Region Graphs

Belief Propagation (BP) [24] is an algorithm for computing marginal probabilities in distributions taking the following form:

\[ p(x) = \frac{1}{Z} \prod_a f_a(x_a) \]  

where \( a \) indexes the factors in the model and \( f(x_a) \) is a function on \( x_a \), which is a subset of \( \{x_1, ..., x_n\} \), the \( n \) discrete-valued random variables in the distribution. In this paper we assume \( p(x) \) to be comprised of only pairwise factors but the extension to factor graphs is straightforward.

BP is a message passing algorithm that computes marginals by iteratively sending messages from one node to its neighboring nodes. It is most easily described in terms of messages passed along the edges of a factor graph [15] (factors are pairwise interactions in this paper), which is a bipartite graph comprised of factor nodes and variable nodes. The edges in a factor graph connect each factor node \( a \) to the variable nodes \( i \) for which \( x_i \in x_a \). The belief \( b_i(x_i) \) at variable node \( i \) is an approximation to the exact marginal \( p(x_i) \) and is defined as:

\[ b_i(x_i) \propto \prod_{a \in N(i)} m_{a \rightarrow i}(x_i) \]  

where \( m_{a \rightarrow i}(x_i) \) is a message from factor node \( a \) to variable node \( i \) and \( N(i) \) is the set of factor nodes neighboring variable node \( i \). The belief \( b_i(x_i) \) over the variables \( x_a \) is defined in an analogous fashion, as a product of messages from variable nodes to factor node \( a \). These beliefs will be exact if the factor graph contains no cycles.

An important limitation of BP is that messages are defined on a single variable. This means that interactions between variables may be lost during message passing. Generalized Belief Propagation (GBP) [34] is a class of algorithms that address this limitation. In GBP, messages over one or more variable are passed among sets of nodes or regions. A region graph is a structure, analogous to the factor graph, that helps organize the computation of GBP messages.

**DEFINITION 1.** A Region \( R \) is a set of variable nodes and factor nodes such that if factor node \( a \) is in region \( R \), all variable nodes \( i \) where \( x_i \in x_a \) are also in \( R \).

Let \( \mathcal{R} \) denote the set of all regions. Let \( x_R \) denote the set of variables in region \( R \) and \( f_R(x_R) = \prod_{a \in R} f_a(x_a) \) be the factors in region \( R \). Every variable and factor must belong to some region.

**DEFINITION 2.** A Region Graph (RG) is a directed graph \( G(V, E) \), where each vertex \( v \in V \) is associated with a region \( R \in \mathcal{R} \) and the directed edges \( e \in E \) are from some vertex \( v_i \) to vertex \( v_j \). If \( x_{R_e} \subseteq x_{R_i}, \text{ where } x_{R_e} \text{ is the set of variables in region } R_e \). In such cases, vertex (region) \( p \) is the parent of vertex \( c \).

Let \( pa(R) \) denote the parents, \( an(R) \) denote the ancestors and \( de(R) \) denote the descendants of region \( R \). Regions with no parents will be referred to as outer regions; all other regions are inner regions. The Bethe RG is a RG with outer regions for each factor and inner regions for each variable.

Each region \( R \) is associated with a counting number \( \kappa_R \) used to define the region-based free energy of a RG. Let \( R(i) = \{ R \in \mathcal{R} | x_i \in x_R \} \) denote the set of regions containing variable \( i \). A RG is considered 1-balanced if:

\[ \sum_{R \in R(i)} \kappa_R = 1 \forall i \]  

1-balancedness is satisfied if \( \kappa_R \) is defined recursively as:

\[ \kappa_R = 1 - \sum_{A \in an(R)} \kappa_A \]  

A RG is 1-Connected if the subgraph consisting of the regions \( R(i) \) is connected for all \( i \). These conditions can be strengthened by considering the set of regions containing a larger set of variables - i.e. \( R(s) = \{ R \in \mathcal{R} | x_s \in x_R \} \) for \( x_s \subseteq \{x_1, ..., x_n\} \). A RG is called totally connected and balanced if it is connected and balanced for all sets of variables that are subsets of an outer region - i.e. for any \( x_s \subseteq x_{R_o} \) for outer region \( R \) [23]. Junction graphs [18] and Join Graphs [3] are two-layer RG constructions satisfying 1-balancedness and 1-connectedness, while CVM [14, 21] ensures total connectivity and balancedness.

Many different GBP algorithms can be defined on RGs. The parent-to-child (or canonical) GBP algorithm is one such algorithm. It is defined in terms of messages sent from a parent region to a child region. As in BP, the belief at a particular region \( R \) is computed as a product of messages