61 Prove the identity
\[ \sum_{k=0}^{n} \frac{1}{F_{2k}} = 3 - \frac{F_{2n-1}}{F_{2n}}, \text{ integer } n \geq 1. \]

What is \( \sum_{k=0}^{n} \frac{1}{F_{2k}} \)?

62 Let \( A_n = \phi^n + \phi^{-n} \) and \( B_n = \phi^n - \phi^{-n} \).
   a Find constants \( \alpha \) and \( \beta \) such that \( A_n = \alpha A_{n-1} + \beta A_{n-2} \) and \( B_n = \alpha B_{n-1} + \beta B_{n-2} \) for all \( n \geq 0 \).
   b Express \( A_n \) and \( B_n \) in terms of \( F_n \) and \( L_n \) (see exercise 28).
   c Prove that \( \sum_{k=1}^{n} \frac{1}{F_{2k+1} + 1} = \frac{B_n}{A_n + 1} \).
   d Find a closed form for \( \sum_{k=1}^{n} \frac{1}{F_{2k+1} - 1} \).

**Bonus problems**

63 How many permutations \( \pi_1 \pi_2 \ldots \pi_n \) of \( \{1, 2, \ldots, n\} \) have exactly \( k \) indices \( i, j \) such that
   a \( \pi_i < \pi_j \) for all \( i < j \)? (Such \( j \) are called “left-to-right maxima!”)
   b \( \pi_i > j \)? (Such \( j \) are called “excedances!”)

64 What is the denominator of \( \frac{1}{1/2 - n} \), when this fraction is reduced to lowest terms?

65 Prove the identity
\[ \int_{0}^{1} \ldots \int_{0}^{1} f([x_1 + \ldots + x_n]) \, dx_1 \ldots dx_n = \sum_{k=0}^{n} \frac{f(k)}{k!}. \]

66 Show that \( \langle \binom{n}{1} \rangle = 2^{\binom{n}{2}} \), and find a closed form for \( \langle \binom{n}{2} \rangle \).

67 Find a closed form for \( \sum_{k=1}^{n} k^2 H_{n+k} \).

68 Show that the generalized harmonic numbers of exercise 22 have the power series expansion
\[ H_z = \sum_{n \geq 2} (-1)^n H_{\infty}^{(n)} z^{n-1}. \]

69 Prove that the generalized factorial of equation (5.83) can be written
\[ \prod_{k \geq 1} \left( 1 + \frac{z}{k} \right) e^{-z/k} = \frac{e^{yz}}{z^y}, \]
by considering the limit as \( n \to \infty \) of the first \( n \) factors of this infinite product. Show that \( \frac{d}{dz} [z!] \) is related to the general harmonic numbers of exercise 22.