80 Show that continuant polynomials appear in the matrix product
\[
\begin{pmatrix}
0 & 1 \\
1 & x_1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & x_2
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
1 & x_n
\end{pmatrix}
\]
and in the determinant
\[
\begin{vmatrix}
\ddots & \ddots & \ddots & \ddots \\
& 1 & x_1 & 1 & 1 & \ldots & 1 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 1 & x_3 & a_1 & \ldots & 1 \\
& & & & \ddots & \ddots & \ddots & \ddots \\
& & & & & 1 & x_{n-1} & a_1 & \ldots & 1
\end{vmatrix}
\]
and in the determinant
\[
\det
\begin{pmatrix}
\begin{pmatrix}
0 & -1 & 0 & \cdots & 0 \\
1 & x_1 & 1 & \cdots & 1 \\
& \ddots & \ddots & \ddots & \ddots \\
& & 1 & x_{n-1} & a_1 & \ldots & 1 \\
& & & \ddots & \ddots & \ddots & \ddots \\
& & & & 1 & x_n & a_1 & \ldots & 1
\end{pmatrix}
\end{pmatrix}
\]

81 Generalizing (6.146), find a continued fraction related to the generating function \( \sum_{n \geq 1} z^{[n \alpha]} \), where \( \alpha \) is any positive irrational number.

82 Let \( m \) and \( n \) be odd, positive integers. Find closed forms for
\[
S_{m,n}^+ = \sum_{k \geq 0} \frac{1}{F_{2mk+n} + F_m}; \quad S_{m,n}^- = \sum_{k \geq 0} \frac{1}{F_{2mk+n} - F_m}.
\]

Hint: The sums in exercise 62 are \( S_{1,3}^+ = S_{1,2n+3}^+ \) and \( S_{1,3}^- = S_{1,2n+3}^- \).

83 Let \( \alpha \) be an irrational number in \( \{0,1\} \) and let \( a_2, a_3, \ldots \) be the partial quotients in its continued fraction representation. Show that \( D(\alpha, n) < 2 \) when \( n = K(\alpha_1, \ldots, \alpha) \), where \( D \) is the discrepancy defined in Chapter 3.

84 Let \( Q_n \) be the largest denominator on level \( n \) of the Stern-Brocot tree. (Thus \( (Q_0, Q_1, Q_2, Q_3, Q_4, \ldots) = (1, 2, 3, 5, 8, \ldots) \) according to the diagram in Chapter 4.) Prove that \( Q_n = F_{n+2} \).

85 Characterize all \( N \) such that the Fibonacci residues
\[
\{F_0 \mod N, F_1 \mod N, F_2 \mod N, \ldots\}
\]
form the complete set \( \{0, 1, \ldots, N - 1\} \). (See exercise 59.)

Research problems

86 What is the best way to extend the definition of \( \binom{n}{k} \) to arbitrary real values of \( n \) and \( k \)?

87 Let \( H_n \) be written in lowest terms as \( a_n/b_n \), as in exercise 52.
   a Are there infinitely many \( n \) with \( 11 \nmid a_n \)?
   b Are there infinitely many \( n \) with \( b_n = \text{lcm}(1, 2, \ldots, n) \)? (Two such values are \( n = 250 \) and \( n = 1000 \).)

88 Prove that \( \gamma \) and \( e^\gamma \) are irrational.