of $f'$ is related in a simple way to the transform of $f$. The relationship is expressed in the following theorem.

**Theorem 6.2.1** Suppose that $f$ is continuous and $f'$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants $K$, $a$, and $M$ such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then $L\{f'(t)\}$ exists for $s > a$, and moreover

$$ L\{f'(t)\} = sL\{f(t)\} - f(0). $$

(1)

To prove this theorem we consider the integral

$$ \int_0^A e^{-st} f'(t) \, dt. $$

If $f'$ has points of discontinuity in the interval $0 \leq t \leq A$, let them be denoted by $t_1, t_2, \ldots, t_n$. Then we can write this integral as

$$ \int_0^A e^{-st} f'(t) \, dt = \int_0^{t_1} e^{-st} f'(t) \, dt + \int_{t_1}^{t_2} e^{-st} f'(t) \, dt + \cdots + \int_{t_n}^A e^{-st} f'(t) \, dt. $$

Integrating each term on the right by parts yields

$$ \int_0^A e^{-st} f'(t) \, dt = e^{-st} f(t) \bigg|_0^{t_1} + e^{-st} f(t) \bigg|_{t_1}^{t_2} + \cdots + e^{-st} f(t) \bigg|_{t_n}^A $$

$$ + s \left[ \int_0^{t_1} e^{-st} f(t) \, dt + \int_{t_1}^{t_2} e^{-st} f(t) \, dt + \cdots + \int_{t_n}^A e^{-st} f(t) \, dt \right]. $$

Since $f$ is continuous, the contributions of the integrated terms at $t_1, t_2, \ldots, t_n$ cancel. Combining the integrals gives

$$ \int_0^A e^{-st} f'(t) \, dt = e^{-sA} f(A) - f(0) + s \int_0^A e^{-st} f(t) \, dt. $$

As $A \to \infty$, $e^{-sA} f(A) \to 0$ whenever $s > a$. Hence, for $s > a$,

$$ L\{f'(t)\} = sL\{f(t)\} - f(0), $$

which establishes the theorem.

If $f'$ and $f''$ satisfy the same conditions that are imposed on $f$ and $f'$, respectively, in Theorem 6.2.1, then it follows that the Laplace transform of $f''$ also exists for $s > a$ and is given by

$$ L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0). $$

(2)

Indeed, provided the function $f$ and its derivatives satisfy suitable conditions, an expression for the transform of the $n$th derivative $f^{(n)}$ can be derived by successive applications of this theorem. The result is given in the following corollary.

**Corollary 6.2.2** Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous, and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants $K, a,$