Generating Functions

THE MOST POWERFUL WAY to deal with sequences of numbers, as far as anybody knows, is to manipulate infinite series that “generate” those sequences. We’ve learned a lot of sequences and we’ve seen a few generating functions; now we’re ready to explore generating functions in depth, and to see how remarkably useful they are.

7.1 DOMINO THEORY AND CHANGE

Generating functions are important enough, and for many of us new enough, to justify a relaxed approach as we begin to look at them more closely. So let’s start this chapter with some fun and games as we try to develop our intuitions about generating functions. We will study two applications of the ideas, one involving dominoes and the other involving coins.

How many ways \( T_n \) are there to completely cover a 2 x \( n \) rectangle with 2 x 1 dominoes? We assume that the dominoes are identical (either because they’re face down, or because someone has rendered them indistinguishable, say by painting them all red); thus only their orientations-vertical or horizontal-matter, and we can imagine that we’re working with domino-shaped tiles. For example, there are three tilings of a 2 x 3 rectangle, namely \( \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \), \( \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \), and \( \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \); so \( T_3 = 3 \).

To find a closed form for general \( T_n \) we do our usual first thing, look at small cases. When \( n = 1 \) there’s obviously just one tiling, \( \begin{array}{ccc} \square \end{array} \); and when \( n = 2 \) there are two, \( \begin{array}{ccc} \square \end{array} \) and \( \begin{array}{ccc} \square \end{array} \).

How about when \( n = 0 \); how many tilings of a 2 x 0 rectangle are there? It’s not immediately clear what this question means, but we’ve seen similar situations before: There is one permutation of zero objects (namely the empty permutation), so \( 0! = 1 \). There is one way to choose zero things from \( n \) things (namely to choose nothing), so \( \binom{n}{0} = 1 \). There is one way to partition the empty set into zero nonempty subsets, but there are no such ways to partition a nonempty set; so \( \binom{n}{0} = [n = 0] \). By such reasoning we can conclude that