(The last step replaces \( k - j \) by \( m \); this is legal because we have \( \binom{k}{j} = 0 \) when \( 0 \leq k < j \).) We conclude that \( \binom{j + m}{j} \) is the number of ways to tile a 2 \( \times (j + 2m) \) rectangle with \( j \) vertical dominoes and \( 2m \) horizontal dominoes. For example, we recently looked at the 2 \( \times 10 \) tiling \( \mathbf{C E E R I R J} \), which involves four verticals and six horizontals; there are \( \binom{4 + 6}{4} = 35 \) such tilings in all, so one of the terms in the commutative version of \( T \) is \( 350406 \).

We can suppress even more detail by ignoring the orientation of the dominoes. Suppose we don’t care about the horizontal/vertical breakdown; we only want to know about the total number of 2 \( \times n \) tilings. (This, in fact, is the number \( T_n \) we started out trying to discover.) We can collect the necessary information by simply substituting a single quantity, \( z \), for \( 0 \) and \( O \). And we might as well also replace \( I \) by 1, getting

\[
T = \frac{1}{1 - z - 2z^2}.
\]  

This is the generating function (6.117) for Fibonacci numbers, except for a missing factor of \( z \) in the numerator; so we conclude that the coefficient of \( z^n \) in \( T \) is \( F_{n+1} \).

The compact representations \( 1/(1-0-R) \), \( 1/(1-0-EI) \), and \( 1/(1-z-z^2) \) that we have deduced for \( T \) are called \textit{generating functions}, because they generate the coefficients of interest.

Incidentally, our derivation implies that the number of 2 \( \times n \) domino tilings with exactly \( m \) pairs of horizontal dominoes is \( \binom{n-m}{m} \). (This follows because there are \( j = n - 2m \) vertical dominoes, hence there are

\[
\binom{j + m}{j} = \binom{j + m}{m} = \binom{n - m}{m}
\]

ways to do the tiling according to our formula.) We observed in Chapter 6 that \( \binom{n-m}{m} \) is the number of Morse code sequences of length \( n \) that contain \( m \) dashes; in fact, it’s easy to see that 2 \( \times n \) domino tilings correspond directly to Morse code sequences. (The tiling \( \mathbf{C E E R I R J} \) corresponds to ‘\( - - \cdots - \)’.) Thus domino tilings are closely related to the continuant polynomials we studied in Chapter 6. It’s a small world.

We have solved the \( T_n \) problem in two ways. The first way, guessing the answer and proving it by induction, was easier; the second way, using infinite sums of domino patterns and distilling out the coefficients of interest, was fancier. But did we use the second method only because it was amusing to play with dominoes as if they were algebraic variables? No; the real reason for introducing the second way was that the infinite-sum approach is a lot more powerful. The second method applies to many more problems, because, it doesn’t require us to make magic guesses.