Recall from Section 5.4 that \( t = 0 \) is a regular singular point for this equation, and therefore solutions may become unbounded as \( t \to 0 \). However, let us try to determine whether there are any solutions that remain finite at \( t = 0 \) and have finite derivatives there. Assuming that there is such a solution \( y = \phi(t) \), let \( Y(s) = \mathcal{L}[\phi(t)] \).

(a) Show that \( Y(s) \) satisfies

\[
(1 + s^2)Y'(s) + sY(s) = 0.
\]

(b) Show that \( Y(s) = c(1 + s^2)^{-1/2} \), where \( c \) is an arbitrary constant.

(c) Expanding \( (1 + s^2)^{-1/2} \), a binomial series valid for \( s > 1 \) and assuming that it is permissible to take the inverse transform term by term, show that

\[
y = c \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n (n!)^2} = c J_0(t),
\]

where \( J_0 \) is the Bessel function of the first kind of order zero. Note that \( J_0(0) = 1 \), and that \( J_0 \) has finite derivatives of all orders at \( t = 0 \). It was shown in Section 5.8 that the second solution of this equation becomes unbounded as \( t \to 0 \).

36. For each of the following initial value problems use the results of Problem 28 to find the differential equation satisfied by \( Y(s) = \mathcal{L}[\phi(t)] \), where \( y = \phi(t) \) is the solution of the given initial value problem.

(a) \( y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 1 \) (Airy’s equation)

(b) \( (1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0; \quad y(0) = 0, \quad y'(0) = 1 \) (Legendre’s equation)

Note that the differential equation for \( Y(s) \) is of first order in part (a), but of second order in part (b). This is due to the fact that \( t \) appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

37. Suppose that

\[
g(t) = \int_0^t f(\tau) d\tau.
\]

If \( G(s) \) and \( F(s) \) are the Laplace transforms of \( g(t) \) and \( f(t) \), respectively, show that

\[
G(s) = \frac{F(s)}{s}.
\]

38. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

\[
F(s) = \frac{P(s)}{Q(s)},
\]

where \( Q(s) \) is a polynomial of degree \( n \) with distinct zeros \( r_1, \ldots, r_n \) and \( P(s) \) is a polynomial of degree less than \( n \). In this case it is possible to show that \( P(s)/Q(s) \) has a partial fraction expansion of the form

\[
\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n},
\]

where the coefficients \( A_1, \ldots, A_n \) must be determined.

(a) Show that

\[
A_k = \frac{P(r_k)}{Q'(r_k)}, \quad k = 1, \ldots, n.
\]

Hint: One way to do this is to multiply Eq. (i) by \( s - r_k \) and then to take the limit as \( s \to r_k \).