for some $k$ and $m$; hence $2k + k + 3m = 6$. In other words, $k + m = 2$.
If we use no verticals, then $k = 0$ and $m = 2$; the number of possibilities is $\binom{2+0}{2}2^0 = 1$. (This accounts for the tiling \[\square\square\square\square\square\].) If we use two verticals, then $k = 1$ and $m = 1$; there are $\binom{1+2}{1}2^1 = 6$ such tilings. And if we use four verticals, then $k = 2$ and $m = 0$; there are $\binom{0+4}{0}2^2 = 4$ such tilings, making a total of $U_4 = 11$. In general if $n$ is even, this reasoning shows that $k + m = \frac{1}{2}n$, hence $\binom{\frac{n}{2}+k}{m} = \binom{n/2+k}{m/2-k}$ and the total number of $3 \times n$ tilings is

$$U_n = \sum \binom{n/2+k}{n/2-k}2^k = \sum \binom{n-m}{m}2^{n/2-m}. \quad (7.9)$$

As before, we can also substitute $z$ for both 0 and 0, getting a generating function that doesn’t discriminate between dominoes of particular persuasions. The result is

$$U = \frac{1 - z^3}{1 - z^2(1 - z^3)^3(1 - z^3)^3} = 1 - 4z^5 + z^6. \quad (7.10)$$

If we expand this quotient into a power series, we get

$$U = 1 + U_2 z^3 + U_4 z^6 + U_6 z^9 + U_8 z^{12} + \cdots,$$

a generating function for the numbers $U_n$. (There’s a curious mismatch between subscripts and exponents in this formula, but it is easily explained. The coefficient of $z^9$, for example, is $U_6$, which counts the tilings of a $3 \times 6$ rectangle. This is what we want, because every such tiling contains nine dominoes.)

We could proceed to analyze (7.10) and get a closed form for the coefficients, but it’s better to save that for later in the chapter after we’ve gotten more experience. So let’s divest ourselves of dominoes for the moment and proceed to the next advertised problem, “change!”

Ah yes, I remember when we had half-dollars.

How many ways are there to pay 50 cents? We assume that the payment must be made with pennies 0, nickels 0, dimes 0, quarters 0, and half-dollars 0. George Polya [239] popularized this problem by showing that it can be solved with generating functions in an instructive way.

Let’s set up infinite sums that represent all possible ways to give change, just as we tackled the domino problems by working with infinite sums that represent all possible domino patterns. It’s simplest to start by working with fewer varieties of coins, so let’s suppose first that we have nothing but pennies. The sum of all ways to leave some number of pennies (but just pennies) in change can be written

$$P = f + 0 + 00 + 000 + 0000 + \cdots = f + 0 + 0^2 + 0^3 + 0^4 + \cdots.$$