7.2 BASIC MANEUVERS

Now let’s look more closely at some of the techniques that make power series powerful.

First a few words about terminology and notation. Our generic generating function has the form

\[ G(z) = g_0 + g_1 z + g_2 z^2 + \cdots = \sum_{n \geq 0} g_n z^n, \]  

and we say that \( G(z) \), or \( G \) for short, is the generating function for the sequence \( (g_0, g_1, g_2, \ldots) \), which we also call \( (g_n) \). The coefficient \( g_n \) of \( z^n \) in \( G(z) \) is sometimes denoted \([z^n] G(z)\).

The sum in (7.12) runs over all \( n \geq 0 \), but we often find it more convenient to extend the sum over all integers \( n \). We can do this by simply regarding \( g_{-1} = g_{-2} = \ldots = 0 \). In such cases we might still talk about the sequence \( (g_0, g_1, g_2, \ldots) \), as if the \( g_n \)'s didn’t exist for negative \( n \).

Two kinds of “closed forms” come up when we work with generating functions. We might have a closed form for \( G(z) \), expressed in terms of \( z \); or we might have a closed form for \( g_n \), expressed in terms of \( n \). For example, the generating function for Fibonacci numbers has the closed form \( z/(1 - z - z^2) \); the Fibonacci numbers themselves have the closed form \( \phi^n - \bar{\phi}^n)/\sqrt{5} \). The context will explain what kind of closed form is meant.

Now a few words about perspective. The generating function \( G(z) \) appears to be two different entities, depending on how we view it. Sometimes it is a function of a complex variable \( z \), satisfying all the standard properties proved in calculus books. And sometimes it is simply a formal power series, with \( z \) acting as a placeholder. In the previous section, for example, we used the second interpretation; we saw several examples in which \( z \) was substituted for some feature of a combinatorial object in a “sum” of such objects. The coefficient of \( z^n \) was then the number of combinatorial objects having \( n \) occurrences of that feature.

When we view \( G(z) \) as a function of a complex variable, its convergence becomes an issue. We said in Chapter 2 that the infinite series \( \sum_{n \geq 0} g_n z^n \) converges (absolutely) if and only if there’s a bounding constant \( A \) such that the finite sums \( \sum_{n \leq N} |g_n z^n| \) never exceed \( A \), for any \( N \). Therefore it’s easy to see that if \( \sum_{n \geq 0} g_n z^n \) converges for some value \( z = z_0 \), it also converges for all \( z \) with \( |z| < |z_0| \). Furthermore, we must have \( \lim_{n \to \infty} |g_n z_0^n| = 0 \); hence, in the notation of Chapter 9, \( g_n = \mathcal{O}(1/|z_0|^n) \) if there is convergence at \( z_0 \). And conversely if \( g_n = \mathcal{O}(M^n) \), the series \( \sum_{n \geq 0} g_n z^n \) converges for all \( |z| < 1/M \). These are the basic facts about convergence of power series.

But for our purposes convergence is usually a red herring, unless we’re trying to study the asymptotic behavior of the coefficients. Nearly every