operation we perform on generating functions can be justified rigorously as an operation on formal power series, and such operations are legal even when the series don’t converge. (The relevant theory can be found, for example, in Bell [19], Niven [225], and Henrici [151, Chapter 1].)

Furthermore, even if we throw all caution to the winds and derive formulas without any rigorous justification, we generally can take the results of our derivation and prove them by induction. For example, the generating function for the Fibonacci numbers converges only when \( |z| < 1/\phi \approx 0.618 \), but we didn’t need to know that when we proved the formula \( F_n = (\phi^n - (1 - \phi)^n) \sqrt{5} \).

The latter formula, once discovered, can be verified directly, if we don’t trust the theory of formal power series. Therefore we’ll ignore questions of convergence in this chapter; it’s more a hindrance than a help.

So much for perspective. Next we look at our main tools for reshaping generating functions—adding, shifting, changing variables, differentiating, integrating, and multiplying. In what follows we assume that, unless stated otherwise, \( F(z) \) and \( G(z) \) are the generating functions for the sequences \( \langle f_n \rangle \) and \( \langle g_n \rangle \). We also assume that the \( f_n \)'s and \( g_n \)'s are zero for negative \( n \), since this saves us some bickering with the limits of summation.

It’s pretty obvious what happens when we add constant multiples of \( F \) and \( G \) together:

\[
\alpha F(z) + \beta G(z) = \alpha \sum_n f_n z^n + \beta \sum_n g_n z^n = \sum_n (\alpha f_n + \beta g_n) z^n.
\] (7.13)

This gives us the generating function for the sequence \( \langle \alpha f_n + \beta g_n \rangle \).

Shifting a generating function isn’t much harder. To shift \( G(z) \) right by \( m \) places, that is, to form the generating function for the sequence \( \langle 0, \ldots, 0, g_0, g_1, \ldots \rangle = \langle g_{n-m} \rangle \) with \( m \) leading 0’s, we simply multiply by \( z^m \):

\[
z^m G(z) = \sum_n g_n z^{n+m} = \sum_n g_{n-m} z^n, \quad \text{integer } m \geq 0.
\] (7.14)

This is the operation we used (twice), along with addition, to deduce the equation \( (1 - z - z^2)F(z) = z \) on our way to finding a closed form for the Fibonacci numbers in Chapter 6.

And to shift \( G(z) \) left \( m \) places—that is, to form the generating function for the sequence \( \langle g_m, g_{m-1}, g_{m+1}, \ldots \rangle = \langle g_{n+m} \rangle \) with the first \( m \) elements discarded— we subtract off the first \( m \) terms and then divide by \( z^m \):

\[
\frac{G(z)g_0 \cdot g_z \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n \geq m} g_n z^{n-m} = \sum_{n \geq 0} g_{n+m} z^n.
\] (7.15)

(We can’t extend this last sum over all \( n \) unless \( g_0 = \ldots = g_{m-1} = 0 \).)