6.4 Differential Equations with Discontinuous Forcing Functions

Find the solution of the differential equation
\[ 2y'' + y' + 2y = g(t), \]  
(1)

where
\[ g(t) = u_5(t) - u_{20}(t) = \begin{cases} 
1, & 5 \leq t < 20, \\
0, & 0 \leq t < 5 \text{ and } t \geq 20. 
\end{cases} \]  
(2)

Assume that the initial conditions are
\[ y(0) = 0, \quad y'(0) = 0. \]  
(3)

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for \( 5 \leq t < 20 \). Alternatively, \( y \) may represent the response of a damped oscillator subject to the applied force \( g(t) \).

The Laplace transform of Eq. (1) is
\[ 2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = (e^{-5s} - e^{-20s})/s. \]

Introducing the initial values (3) and solving for \( Y(s) \), we obtain
\[ Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \]  
(4)

To find \( y = \phi(t) \) it is convenient to write \( Y(s) \) as
\[ Y(s) = (e^{-5s} - e^{-20s})H(s), \]  
(5)

where
\[ H(s) = 1/s(2s^2 + s + 2). \]  
(6)

Then, if \( h(t) = \mathcal{L}^{-1}\{H(s)\} \), we have
\[ y = \phi(t) = u_5(t)h(t - 5) - u_{20}(t)h(t - 20). \]  
(7)