7.3 SOLVING RECURRENCES

Now let's focus our attention on one of the most important uses of generating functions: the solution of recurrence relations.

Given a sequence \( \langle a_n \rangle \) that satisfies a given recurrence, we seek a closed form for \( a_n \) in terms of \( n \). A solution to this problem via generating functions proceeds in four steps that are almost mechanical enough to be programmed on a computer:

1. Write down a single equation that expresses \( a_n \) in terms of other elements of the sequence. This equation should be valid for all integers \( n \), assuming that \( a_{-1} = a_{-2} = \ldots = 0 \).

2. Multiply both sides of the equation by \( z^n \) and sum over all \( n \). This gives, on the left, the sum \( \sum_n a_n z^n \), which is the generating function \( G(z) \). The right-hand side should be manipulated so that it becomes some other expression involving \( G(z) \).

3. Solve the resulting equation, getting a closed form for \( G(z) \).

4. Expand \( G(z) \) into a power series and read off the coefficient of \( z^n \); this is a closed form for \( a_n \).

This method works because the single function \( G(z) \) represents the entire sequence \( \langle a_n \rangle \) in such a way that many manipulations are possible.

Example 1: Fibonacci numbers revisited.

For example, let's rerun the derivation of Fibonacci numbers from Chapter 6. In that chapter we were feeling our way, learning a new method; now we can be more systematic. The given recurrence is

\[
\begin{align*}
g_0 &= 0; \\
g_1 &= 1; \\
g_n &= g_{n-1} + g_{n-2}, \quad \text{for } n \geq 2.
\end{align*}
\]

We will find a closed form for \( g_n \) by using the four steps above.

Step 1 tells us to write the recurrence as a "single equation" for \( g_n \). We could say

\[
g_n = \begin{cases} 
0, & \text{if } n \leq 0; \\
1, & \text{if } n = 1; \\
g_{n-1} + g_{n-2}, & \text{if } n > 1;
\end{cases}
\]

but this is cheating. Step 1 really asks for a formula that doesn't involve a case-by-case construction. The single equation

\[
g_n = g_{n-1} + g_{n-2}
\]

works for \( n \geq 2 \) and it also holds when \( n \leq 0 \) (because we have \( g_0 = 0 \) and \( g_{\text{negative}} = 0 \)). But when \( n = 1 \) we get 1 on the left and 0 on the right.