While it may be helpful to visualize the solution shown in Figure 6.4.1 as composed of solutions of three separate initial value problems in three separate intervals, it is somewhat tedious to find the solution by solving these separate problems. Laplace transform methods provide a much more convenient and elegant approach to this problem and to others having discontinuous forcing functions.

The effect of the discontinuity in the forcing function can be seen if we examine the solution $\phi(t)$ of Example 1 more closely. According to the existence and uniqueness Theorem 3.2.1 the solution $\phi$ and its first two derivatives are continuous except possibly at the points $t = 5$ and $t = 20$ where $g$ is discontinuous. This can also be seen at once from Eq. (7). One can also show by direct computation from Eq. (7) that $\phi$ and $\phi'$ are continuous even at $t = 5$ and $t = 20$. However, if we calculate $\phi''$, we find that

$$\lim_{t \to 5^-} \phi''(t) = 0, \quad \lim_{t \to 5^+} \phi''(t) = 1/2.$$

Consequently, $\phi''(t)$ has a jump of $1/2$ at $t = 5$. In a similar way one can show that $\phi''(t)$ has a jump of $-1/2$ at $t = 20$. Thus the jump in the forcing term $g(t)$ at these points is balanced by a corresponding jump in the highest order term $2y''$ on the left side of the equation.

Consider now the general second order linear equation

$$y'' + p(t)y' + q(t)y = g(t), \quad (15)$$

where $p$ and $q$ are continuous on some interval $\alpha < t < \beta$, but $g$ is only piecewise continuous there. If $y = \psi(t)$ is a solution of Eq. (15), then $\psi$ and $\psi'$ are continuous on $\alpha < t < \beta$, but $\psi''$ has jump discontinuities at the same points as $g$. Similar remarks apply to higher order equations; the highest derivative of the solution appearing in the differential equation has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous even at those points.