EXAMPLE 2

Describe the qualitative nature of the solution of the initial value problem

\[ y'' + 4y = g(t), \quad (16) \]

\[ y(0) = 0, \quad y'(0) = 0, \quad (17) \]

where

\[ g(t) = \begin{cases} 
0, & 0 \leq t < 5, \\
(t - 5)/5, & 5 \leq t < 10, \\
1, & t \geq 10, 
\end{cases} \quad (18) \]

and then find the solution.

In this example the forcing function has the graph shown in Figure 6.4.2 and is known as ramp loading. It is relatively easy to identify the general form of the solution. For \( t < 5 \) the solution is simply \( y = 0 \). On the other hand, for \( t > 10 \) the solution has the form

\[ y = c_1 \cos 2t + c_2 \sin 2t + 1/4. \quad (19) \]

The constant \( 1/4 \) is a particular solution of the nonhomogeneous equation while the other two terms are the general solution of the corresponding homogeneous equation. Thus the solution (19) is a simple harmonic oscillation about \( y = 1/4 \). Similarly, in the intermediate range \( 5 < t < 10 \), the solution is an oscillation about a certain linear function. In an engineering context, for example, we might be interested in knowing the amplitude of the eventual steady oscillation.

To solve the problem it is convenient to write

\[ g(t) = [u_5(t)(t - 5) - u_{10}(t)(t - 10)]/5, \quad (20) \]

as you may verify. Then we take the Laplace transform of the differential equation and use the initial conditions, thereby obtaining

\[ (s^2 + 4)Y(s) = (e^{-5s} - e^{-10s})/5s^2, \]

or

\[ Y(s) = (e^{-5s} - e^{-10s})H(s)/5, \quad (21) \]

where

\[ H(s) = 1/s^2(s^2 + 4). \quad (22) \]