also has nice coefficients,
\[ [z^n] S(z) = a_1 \left( \frac{m_1 + n}{m_1} \right) \rho_1^n + a_2 \left( \frac{m_2 + n}{m_2} \right) \rho_2^n + \ldots + a_l \left( \frac{m_l + n}{m_l} \right) \rho_l^n, \]  
(7.27)

We will show that every rational function \( R(z) \) such that \( R(0) \neq \infty \) can be expressed in the form
\[ R(z) = S(z) + T(z), \]  
(7.28)
where \( S(z) \) has the form (7.26) and \( T(z) \) is a polynomial. Therefore there is a closed form for the coefficients \([z^n] R(z)\). Finding \( S(z) \) and \( T(z) \) is equivalent to finding the “partial fraction expansion” of \( R(z) \).

Notice that \( S(z) = \infty \) when \( z \) has the values \( 1/\rho_1, \ldots, 1/\rho_l \). Therefore the numbers \( \rho_k \) that we need to find, if we’re going to succeed in expressing \( R(z) \) in the desired form \( S(z) + T(z) \), must be the reciprocals of the numbers \( \alpha_k \) where \( Q(\alpha_k) = 0 \). (Recall that \( R(z) = P(z)/Q(z) \), where \( P \) and \( Q \) are polynomials; we have \( R(z) = \infty \) only if \( Q(z) = 0 \).)

Suppose \( Q(z) \) has the form
\[ Q(z) = q_0 + q_1 z + \cdots + q_m z^m, \]  
where \( q_0 \neq 0 \) and \( q_m \neq 0 \).

The “reflected” polynomial
\[ Q^R(z) = q_0 z^n + q_1 z^{n-1} + \cdots + q_m \]
has an important relation to \( Q(z) \):
\[ Q^R(z) = q_0 (z - \rho_1) \ldots (z - \rho_m) \iff Q(z) = q_0 (1 - \rho_1 z) \ldots (1 - \rho_m z) \]
Thus, the roots of \( Q^R \) are the reciprocals of the roots of \( Q \), and vice versa. We can therefore find the numbers \( \rho_k \) we seek by factoring the reflected polynomial \( Q^R(z) \).

For example, in the Fibonacci case we have
\[ Q(z) = 1 - z - z^2; \quad Q^R(z) = z^2 - z - 1. \]
The roots of \( Q^R \) can be found by setting \( (a, b, c) = (1, -1, -1) \) in the quadratic formula \( -b \pm \sqrt{b^2 - 4ac}/2a \); we find that they are
\[ \phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \bar{\phi} = \frac{1 - \sqrt{5}}{2} \]
Therefore \( Q^R(z) = (z - \phi)(z - \bar{\phi}) \) and \( Q(z) = (1 - \phi z)(1 - \bar{\phi} z) \).