(a) Sketch the graph of $f_k(t)$. Observe that the area under the graph is independent of $k$. If $f_k(t)$ represents a force, this means that the product of the magnitude of the force and the time interval during which it acts does not depend on $k$.

(b) Write $f_k(t)$ in terms of the unit step function and then solve the given initial value problem.

(c) Plot the solution for $k = 2$, $k = 1$, and $k = \frac{1}{2}$. Describe how the solution depends on $k$.

Resonance and Beats. In Section 3.9 we observed that an undamped harmonic oscillator (such as a spring–mass system), with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In Problems 19 through 23 we explore the effect of some nonsinusoidal periodic forcing functions.

19. Consider the initial value problem $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, where

$$f(t) = u_0(t) + 2 \sum_{k=1}^{n} (-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $f(t)$ on an interval such as $0 \leq t \leq 6\pi$.

(b) Find the solution of the initial value problem.

(c) Let $n = 15$ and plot the graph of the solution for $0 \leq t \leq 60$. Describe the solution and explain why it behaves as it does.

(d) Investigate how the solution changes as $n$ increases. What happens as $n \to \infty$?

20. Consider the initial value problem $y'' + 0.1y' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, where $f(t)$ is the same as in Problem 19.

(a) Plot the graph of the solution. Use a large enough value of $n$ and a long enough $t$-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

(b) Estimate the amplitude and frequency of the steady-state part of the solution.

(c) Compare the results of part (b) with those from Section 3.9 for a sinusoidally forced oscillator.

21. Consider the initial value problem $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 0$, where

$$g(t) = u_0(t) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $g(t)$ on an interval such as $0 \leq t \leq 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).

(b) Find the solution of the initial value problem.

(c) Let $n = 15$ and plot the graph of the solution for $0 \leq t \leq 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.

(d) Investigate how the solution changes as $n$ increases. What happens as $n \to \infty$?

22. Consider the initial value problem $y'' + 0.1y' + y = g(t)$, $y(0) = 0$, $y'(0) = 0$, where $g(t)$ is the same as in Problem 21.