Step 1 is easy, since we merely need to insert fudge factors to fix things when \( n < 2 \): The equation

\[
g_n = g_{n-1} + 2g_{n-2} + (-1)^n[n \geq 0] + [n = 1]
\]

holds for all integers \( n \). Now we can carry out Step 2:

\[
G(z) = \sum g_n z^n = \sum g_{n-1} z^n + 2 \sum a_n z^n + \sum (n) z^n + \sum z^n = zG(z) + 2z^2G(z) + \frac{1}{1 + z} + z.
\]

(Incidentally, we could also have used \( (-1)^n \) instead of \( (-1)^n \) \( [n \geq 0] \), thereby getting \( \sum \binom{n}{r} z^n = (1 + z)^{-1} \) by the binomial theorem.) Step 3 is elementary algebra, which yields

\[
G(z) = \frac{1 + z(1 + 2z)}{(1 + z)(1 - z - 2z^2)} = \frac{1 + z + z^2}{(1 - 2z)(1 + z)^2}.
\]

And that leaves us with Step 4.

The squared factor in the denominator is a bit troublesome, since we know that repeated roots are more complicated than distinct roots; but there it is. We have two roots, \( \rho_1 = 2 \) and \( \rho_2 = -1 \); the general expansion theorem (7.30) tells us that

\[
g_n = a_1 2^n + (a_2 n + c)(-1)^n
\]

for some constant \( c \), where

\[
a_1 = \frac{1 + 1/2 + 1/4}{(1 + 1/2)^2} = \frac{7}{9}; \quad a_2 = \frac{1 - 1 + 1}{1 - 2/(-1)} = \frac{1}{3}.
\]

(The second formula for \( a_k \) in (7.31) is easier to use than the first one when the denominator has nice factors. We simply substitute \( z = 1/\rho_k \) everywhere in \( R(z) \), except in the factor \( where \) this gives zero, and divide by \( \{d_k - 1\} \); this gives the coefficient of \( n^{d_k - \rho_k^k} \).) Plugging in \( n = 0 \) tells us that the value of the remaining constant \( c \) had better be \( \frac{7}{9} \); hence our answer is

\[
g_n = \frac{7}{9} 2^n + \left(\frac{1}{3} n + \frac{1}{3}\right)(-1)^n.
\]

(7.33)

It doesn’t hurt to check the cases \( n = 1 \) and 2, just to be sure that we didn’t foul up. Maybe we should even try \( n = 3 \), since this formula looks weird. But it’s correct, all right.

Could we have discovered (7.33) by guesswork? Perhaps after tabulating a few more values we may have observed that \( g_{n+1} \approx 2g_n \) when \( n \) is large.