In particular, let us suppose that $t_0$ is zero, and that $g(t)$ is given by

\[ g(t) = d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau, \\ 0, & t \leq -\tau \text{ or } t \geq \tau, \end{cases} \]  

(4)

where $\tau$ is a small positive constant (see Figure 6.5.1). According to Eq. (2) or (3) it follows immediately that in this case $I(\tau) = 1$ independent of the value of $\tau$, as long as $\tau \neq 0$. Now let us idealize the forcing function $d_\tau$ by prescribing it to act over shorter and shorter time intervals; that is, we require that $\tau \to 0$, as indicated in Figure 6.5.2.

As a result of this limiting operation we obtain

\[ \lim_{\tau \to 0} d_\tau(t) = 0, \quad t \neq 0. \]  

(5)

Further, since $I(\tau) = 1$ for each $\tau \neq 0$, it follows that

\[ \lim_{\tau \to 0} I(\tau) = 1. \]  

(6)

Equations (5) and (6) can be used to define an idealized unit impulse function $\delta$, which imparts an impulse of magnitude one at $t = 0$, but is zero for all values of $t$ other than zero. That is, the “function” $\delta$ is defined to have the properties

\[ \delta(t) = 0, \quad t \neq 0; \]  

(7)

\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1. \]  

(8)