is somehow involved. So we steer toward that kind of sum:

\[
G(z) = 1 + \sum_{n} (n+1)g_{n} z^{n+1} = 1 + \sum_{n} ng_{n} z^{n+1} + \sum_{n} g_{n} z^{n+1} = 1 + zG'(z) + zG(z).
\]

Let's check this equation, using the values of \(g_{n}\) for small \(n\). Since

\[
G = 1 + z + 2z^{2} + 6z^{3} + 24z^{4} + \cdots,
\]

\[
G' = 1 + 4z + 18z^{2} + 96z^{3} + \cdots,
\]

we have

\[
z^{2}G' = z^{2} + 4z^{3} + 18z^{4} + 96z^{5} + \cdots,
\]

\[
zG = z + z^{2} + 2z^{3} + 6z^{4} + 24z^{5} + \cdots,
\]

\[
1 = 1.
\]

These three lines add up to \(G\), so we're fine so far. Incidentally, we often find it convenient to write 'G' instead of 'G(z)'; the extra '(z)' just clutters up the formula when we aren't changing \(z\).

Step 3 is next, and it's different from what we've done before because we have a differential equation to solve. But this is a differential equation that we can handle with the hypergeometric series techniques of Section 5.6; those techniques aren't too bad. (Readers who are unfamiliar with hypergeometrics needn't worry- this will be quick.)

First we must get rid of the constant '1', so we take the derivative of both sides:

\[
G' = (z^{2}G' + zG + 1)' = (2zG' + z^{2}G'') + (G + zG') = z^{2}G'' + 3zG' + G.
\]

The theory in Chapter 5 tells us to rewrite this using the 4 operator, and we know from exercise 6.13 that

\[
\delta G = zG', \quad \delta^{2} G = z^{2}G'' + zG'.
\]

Therefore the desired form of the differential equation is

\[
\delta G = z\delta^{2} G + 2z\delta G + zG = z(\delta + 1)^{2} G.
\]

According to (5.109), the solution with \(g_{0} = 1\) is the hypergeometric series \(F(1,1;\cdot;z)\).