Step 3 was more than we bargained for; but now that we know what the function \( G \) is, Step 4 is easy—the hypergeometric definition (5.76) gives us the power series expansion:

\[
G(z) = \, _2F_1 \left( \frac{1}{2}, 1 \mid z \right) = \sum_{n \geq 0} \frac{\Gamma(n+2) n! z^n}{n!} = \sum_{n \geq 0} n! z^n
\]

We’ve confirmed the closed form we knew all along, \( g_n = n! \).

Notice that the technique gave the right answer even though \( G(z) \) diverges for all nonzero \( z \). The sequence \( n! \) grows so fast, the terms \( \frac{n!}{z^n} \) approach \( \infty \) as \( n \to \infty \), unless \( z = 0 \). This shows that formal power series can be manipulated algebraically without worrying about convergence.

**Example 6: A recurrence that goes all the way back.**

Let’s close this section by applying generating functions to a problem in graph theory. A fan of order \( n \) is a graph on the vertices \( \{0, 1, \ldots, n\} \) with \( 2n - 1 \) edges defined as follows: Vertex 0 is connected by an edge to each of the other \( n \) vertices, and vertex \( k \) is connected by an edge to vertex \( k + 1 \), for \( 1 \leq k < n \). Here, for example, is the fan of order 4, which has five vertices and seven edges.

![Diagram of a fan of order 4]

The problem of interest: How many spanning trees \( f_n \) are in such a graph? A spanning tree is a subgraph containing all the vertices, and containing enough edges to make the subgraph connected yet not so many that it has a cycle. It turns out that every spanning tree of a graph on \( n + 1 \) vertices has exactly \( n \) edges. With fewer than \( n \) edges the subgraph wouldn’t be connected, and with more than \( n \) it would have a cycle; graph theory books prove this.

There are \( \binom{2n-1}{n} \) ways to choose \( n \) edges from among the \( 2n - 1 \) present in a fan of order \( n \), but these choices don’t always yield a spanning tree. For instance the subgraph

![Diagram of a subgraph with four edges]

has four edges but is not a spanning tree; it has a cycle from 0 to 4 to 3 to 0, and it has no connection between \( \{1, 2\} \) and the other vertices. We want to count how many of the \( \binom{2n-1}{n} \) choices actually do yield spanning trees.