Turning now to the proof of Theorem 6.6.1, we note first that if

\[ F(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi \]

and

\[ G(s) = \int_0^\infty e^{-s\eta} g(\eta)\,d\eta, \]

then

\[ F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\int_0^\infty e^{-s\eta} g(\eta)\,d\eta. \tag{8} \]

Since the integrand of the first integral does not depend on the integration variable of the second, we can write \( F(s)G(s) \) as an iterated integral,

\[ F(s)G(s) = \int_0^\infty g(\eta)\,d\eta \int_0^\infty e^{-s(\xi+\eta)} f(\xi)\,d\xi. \tag{9} \]

This expression can be put into a more convenient form by introducing new variables of integration. First let \( \xi = t - \eta \), for fixed \( \eta \). Then the integral with respect to \( \xi \) in Eq. (9) is transformed into one with respect to \( t \); hence

\[ F(s)G(s) = \int_0^\infty g(\eta)\,d\eta \int_0^\infty e^{-st} f(t-\eta)\,dt. \tag{10} \]

Next let \( \eta = \tau \); then Eq. (10) becomes

\[ F(s)G(s) = \int_0^\infty g(\tau)\,d\tau \int_\tau^\infty e^{-st} f(t-\tau)\,dt. \tag{11} \]

The integral on the right side of Eq. (11) is carried out over the shaded wedge-shaped region extending to infinity in the \( t\tau \)-plane shown in Figure 6.6.1. Assuming that the order of integration can be reversed, we finally obtain

\[ F(s)G(s) = \int_0^\infty e^{-st}\,dt \int_0^t f(t-\tau)g(\tau)\,d\tau, \tag{12} \]

FIGURE 6.6.1 Region of integration in \( F(s)G(s) \).