Hence
\[ F(z)^2 = \frac{1}{5} \sum_{n \geq 0} (n + 1)(2F_{n+1} - F_n)z^n - \frac{2}{5} \sum_{n \geq 0} F_{n+1}z^n, \]
and we have the answer we seek:
\[ \sum_{k=0}^{n} F_k F_{n-k} = \frac{2nF_{n+1} - (n + 1)F_n}{5} \quad (7.60) \]

For example, when \( n = 3 \) this formula gives \( F_0 F_3 + F_1 F_2 + F_2 F_1 + F_3 F_0 = 0 + 1 + 1 + 0 = 2 \) on the left and \( (6F_4 - 4F_3)/5 = (18 - 8)/5 = 2 \) on the right.

**Example 2: Harmonic convolutions.**

The efficiency of a certain computer method called “samplesort” depends on the value of the sum
\[ T_{m,n} = \sum_{0 \leq k \leq n} \binom{n}{m} \frac{1}{n-k}, \quad \text{integers } m, n \geq 0. \]

Exercise 5.58 obtains the value of this sum by a somewhat intricate double induction, using summation factors. It’s much easier to realize that \( T_{m,n} \) is just the \( n \)th term in the convolution of \( \left( \binom{0}{m}, \binom{1}{m}, \binom{2}{m}, \ldots \right) \) with \( \left( 0, \frac{1}{1}, \frac{1}{2}, \ldots \right) \).

Both sequences have simple generating functions in Table 321:
\[ z^m z^n = \frac{z^m}{(1 - z)^{m+1}}, \quad z^n = \ln \frac{1}{1 - z}. \]

Therefore, by (7.43),
\[ T_{m,n} = [z^n] \frac{z^n}{(1 - z)^{m+1}} \ln \frac{1}{1 - z} = \frac{1}{(1 - z)^{m+1}} \ln \frac{1}{1 - z} \]
\[ = \binom{H_n - H_m}{n - m}. \]

In fact, there are many more sums that boil down to this same sort of convolution, because we have
\[ \frac{1}{(1 - z)^{r+1}} \ln \frac{1}{1 - z} \cdot \frac{1}{(1 - z)^{s+1}} = \frac{1}{(1 - z)^{r+s+2}} \ln \frac{1}{1 - z} \]
for all \( r \) and \( s \). Equating coefficients of \( z^n \) gives the general identity
\[ \sum_{k} \binom{r + k}{k} \binom{s + n - k}{n - k} (H_{r+k} - H_r) = \binom{r + s + n + 1}{n} (H_{r+s+n+1} - H_{r+s+1}) \quad (7.61) \]