for small $\rho$, where $\Omega$ is a domain in $\mathbb{R}^{n-1}$ and $\delta$ is a positive function on $\mathbb{R}_+$ such that $\rho^j\delta^{(j)}(\rho) = o(\delta(\rho))$, $j \geq 1$, and $\delta(\rho) \leq \delta_0$, where $\delta_0$ is a sufficiently small number. Furthermore, we suppose that $L$ is an elliptic differential operator of order $2m$ in $\bar{G}$ with smooth coefficients satisfying Gårding's inequality

$$\text{Re} (Lu, u)_{L^2(\bar{G})} \geq c \|u\|^2_{W^{2m}(\bar{G})}, \quad u \in W^{2m}_0(\bar{G}).$$

Let $V^{l}_{2, \beta, \gamma}(G)$ be the weighted Sobolev space with the norm

$$\|u\|_{V^{l}_{2, \beta, \gamma}(G)} = \left( \sum_{|\alpha| \leq l} \int_{\bar{G}} |x|^{2(\gamma - l + |\alpha|)} \theta^{2(\beta - l + |\alpha|)} |D_x^\alpha u|^2 \, dx \right)^{1/2},$$

where $\theta$ denotes the angle between the vector $0x$ and the $x_n$-axis.

Then as a consequence of Theorem 10.5.4 the following statement holds:

1) The operator of the Dirichlet problem for the equation $Lu = f$ realizes an isomorphism

$$V^{2m}_{2, \beta, \gamma}(G) \rightarrow V^{0}_{2, \beta, \gamma}(G) \times \prod_{k=1}^{m} V^{2m-k+1/2}_{2, \beta, \gamma}(\partial G)$$

if $2m < n - 1$, $\beta \in (2m - (n - 1)/2, (n - 1)/2)$, and $\gamma \in (2m - n/2, n/2)$.

2) If $u \in W^{m}_2(G)$ is the generalized solution of the Dirichlet problem and $f \in V^{0}_{2, \beta, \gamma}(G)$, $g_k \in V^{2m-k+1/2}_{2, \beta, \gamma}(\partial G)$, where $2m < n - 1$, $\beta \in (2m - (n - 1)/2, (n - 1)/2)$, and $\gamma \in (2m - n/2, n/2)$, then $u \in V^{2m}_{2, \beta, \gamma}(G)$.

A large part of this chapter (Sections 10.1 - 10.4) is devoted to the study of general boundary value problems in a cylinder $C$ from which one has removed a tube $D$ with an infinitely small section at infinity. For these problems we construct the inverse operators ”pasting together” the inverse operators of two ”limit problems”. This allows to obtain the Fredholm property for the operator of the boundary value problem in a domain which coincides with the domain $C \setminus D$ for large $x_n$.

Other types of the considered domains can be reduced to this one by a change of variables. Problems in such domains are considered in the last subsection of Section 10.5.

10.1. Formulation of the problem

In this section we introduce the boundary value problem which will be studied in the following sections. We present the so-called auxiliary problem and the limit problems which are closely related to the starting problem. One of the limit problems arises in the cylinder after the freezing at infinity of the coefficients of the initial operators; in this case the tube is replaced by the $x_n$-axis and the boundary conditions given on the boundary of the tube are not taken into account. The second limit problem is a problem in the exterior of the cylinder obtained by ”extension” of the tube. In this case the boundary conditions on the initial cylinder are disregarded, the coefficients of the elliptic equation are replaced by their limiting values at the point of infinity of the $x_n$-axis.
10.1.1. Assumptions on the domain and on the differential operators.

The domain \( \mathcal{G} \). In the sequel, \( l \) is always an integer number \( l \geq 2m \), \( l > \max \mu_{s,k} \). We set \( l_0 = l + \max (0, \max \tau_{s,j}) \). (Here \( \mu_{s,k}, \tau_{s,j} \) are the integer numbers determining the orders of the operators \( B_{s,k}, C_{s,j} \), which will be introduced below.) Let \( \Omega, \hat{\Omega} \) be bounded domains in \( \mathbb{R}^{n-1} \) with boundaries of the class \( C^{l_0} \) such that the origin 0 is contained both in \( \Omega \) and \( \hat{\Omega} \). Furthermore, let \( \varphi \) be a positive function on \( \mathbb{R} \) with \( l_0 \) continuous derivatives such that

\[
(10.1.1) \quad \varphi(t)^{-1} |\varphi^{(j)}(t)| \leq c < +\infty \quad \text{for } j = 1, 2, \ldots, l_0
\]

and \( \varphi(t) \leq \varphi_0 \), where \( \varphi_0 \) is a sufficiently small constant. We set

\[
\mathcal{C} = \{ x = (x', x_n) \in \mathbb{R}^n : x' \in \Omega, x_n \in \mathbb{R} \},
\]

\[
\mathcal{D} = \{ x = (x', x_n) \in \mathbb{R}^n : x_n \in \mathbb{R}, x'/\varphi(x_n) \in \hat{\Omega} \}.
\]

Suppose that \( \mathcal{G} \) is a domain in \( \mathbb{R}^n \) with boundary \( \partial \mathcal{G} \) of class \( C^{l_0} \) such that the set \( \{ x \in \mathcal{G} : x_n < t \} \) is bounded for all \( t \) and

\[
\{ x \in \mathcal{G} : x_n > T \} = \{ x \in \mathcal{C}\backslash \mathcal{D} : x_n > T \}
\]

for sufficiently large \( T \).

The boundary value problem in \( \mathcal{G} \). We denote the connected components of the boundary \( \partial \mathcal{G} \) by \( \Gamma_s, s = 1, \ldots, q \), and consider the boundary value problem

\[
L(x, \partial x) u = f \quad \text{in } \mathcal{G},
\]

\[
B_s(x, \partial x) u + C_s(x, \partial x) u^{(s)} = \bar{g}^{(s)} \quad \text{on } \Gamma_s, \quad s = 1, \ldots, q.
\]

Here \( L \) is a uniformly elliptic differential operator of order \( 2m \), \( B_s \) is a vector of differential operators \( B_{s;k}, k = 1, \ldots, m + j \), \( \text{ord } B_{s;k} \leq \mu_{s,k} \) and \( C_s \) is a matrix of differential operator \( C_{s;k,j} \) which are tangential on \( \Gamma_s \), \( \text{ord } C_{s;k,j} \leq \mu_{s,k} + \tau_{s,j} \), such that problem \( 10.1.2 \), \( 10.1.3 \) is elliptic.

We assume that for arbitrary \( T < \infty \) the coefficients of \( L \) have continuous derivatives up to order \( l - 2m \) in \( \{ x \in \mathcal{G} : x_n < T \} \), while the coefficients of \( B_{s;k} \) and \( C_{s;k,j} \) belong to the class \( C^{l-\mu_{s,k}} \) in a neighbourhood of the set \( \{ x \in \Gamma_s : x_n < T \} \).

Moreover, we assume that the coefficients of \( L \), \( B_{s,k} \), and \( C_{s;k,j} \) satisfy some stabilization condition for large \( x_n \). In order to introduce this condition, we need the following definition.

**Definition 10.1.1.** Let \( G \) be an open subset of the Euclidean space \( \mathbb{R}^n \). The operator

\[
P(x, r \partial x) = \sum_{|\alpha| \leq \mu} p_\alpha(x) (r \partial x)^\alpha
\]

belongs to the class \( \mathcal{O}_k^\mu(G) \) if

\[
|P|_{\mathcal{O}_k^\mu(G)} \overset{\text{def}}{=} \sum_{|\alpha| \leq \mu} \sum_{|\gamma| \leq k} \| (r \partial x)^\gamma p_\alpha \|_{L_\infty(G)} < \infty.
\]

Here \( r = |x'| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \).

In the following, \( T \) is a sufficiently large positive number. Furthermore, let \( \mathcal{U} \) be a cylindrical neighbourhood of the surface \( \partial \mathcal{C} \) and \( \mathcal{V} \) a set of points \( x \in \mathbb{R}^n \) for which \( x'/\varphi(x_n) \) belongs to some neighbourhood of the surface \( \partial \hat{\Omega} \). The sets