We can apply these observations to the spans-of-fans problem considered earlier (Example 6 in Section 7.3). It turns out that there's another way to compute $f_n$, the number of spanning trees of an $n$-fan, based on the configurations of tree edges between the vertices $\{1, 2, \ldots, n\}$: The edge between vertex $k$ and vertex $k + 1$ may or may not be selected for the subtree; and each of the ways to select these edges connects up certain blocks of adjacent vertices. For example, when $n = 10$ we might connect vertices $\{1, 2\}$, $\{3\}$, $\{4, 5, 6, 7\}$, and $\{8, 9, 10\}$:

How many spanning trees can we make, by adding additional edges to vertex $0$? We need to connect $0$ to each of the four blocks; and there are two ways to join $0$ with $\{1, 2\}$, one way to join it with $\{3\}$, four ways with $\{4, 5, 6, 7\}$, and three ways with $\{8, 9, 10\}$, or $2 \times 1 \times 4 \times 3 = 24$ ways altogether. Summing over all possible ways to make blocks gives us the following expression for the total number of spanning trees:

$$f_n = \sum_{m>0} \sum_{k_1+k_2+\ldots+k_n=n} k_1k_2\ldots k_m. \quad (7.64)$$

For example, $f_4 = 4 + 3\cdot1 + 2\cdot2 + 1\cdot3 + 2\cdot1\cdot1 + 1\cdot2\cdot1 + 1\cdot1\cdot2 + 1\cdot1\cdot1 = 21$.

This is the sum of $m$-fold convolutions of the sequence $(0, 1, 2, 3, \ldots)$, for $m = 1, 2, 3, \ldots$; hence the generating function for $(f_n)$ is

$$F(z) = G(z) + G(z)^2 + G(z)^3 + \cdots = \frac{G(z)}{1 - G(z)}$$

where $G(z)$ is the generating function for $(0, 1, 2, 3, \ldots)$, namely $z/(1 - z)^2$. Consequently we have

$$F(z) = \frac{z}{(1-z)^2-z} = \frac{z}{1-3z+z^2},$$

as before. This approach to $(f_n)$ is more symmetrical and appealing than the complicated recurrence we had earlier.