Example 4: A convoluted recurrence.

Our next example is especially important; in fact, it's the "classic example" of why generating functions are useful in the solution of recurrences.

Suppose we have \( n + 1 \) variables \( x_0, x_1, \ldots, x_n \) whose product is to be computed by doing \( n \) multiplications. How many ways \( C_n \) are there to insert parentheses into the product \( x_0 \cdot x_1 \cdot \ldots \cdot x_n \) so that the order of multiplication is completely specified? For example, when \( n = 2 \) there are two ways, \( x_0 \cdot (x_1 \cdot x_2) \) and \( (x_0 \cdot x_1) \cdot x_2 \). And when \( n = 3 \) there are five ways,

\[
\begin{align*}
&x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)), \quad x_0 \cdot [(x_1 \cdot x_2) \cdot x_3], \quad (x_0 \cdot x_1) \cdot (x_2 \cdot x_3), \\
&\quad (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3, \quad ((x_0 \cdot x_1) \cdot x_2) \cdot x_3
\end{align*}
\]

Thus \( C_2 = 2 \), \( C_3 = 5 \); we also have \( C_1 = 1 \) and \( C_0 = 1 \).

Let's use the four-step procedure of Section 7.3. What is a recurrence for the \( C_n \)'s? The key observation is that there's exactly one operation outside all of the parentheses, when \( n > 0 \); this is the final multiplication that ties everything together. If this operation occurs between \( x_k \) and \( x_{k+1} \), there are \( C_k \) ways to fully parenthesize \( x_0, \ldots, x_k \) and there are \( C_{n-k-1} \) ways to fully parenthesize \( x_{k+1}, \ldots, x_n \); hence

\[
C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0, \quad \text{if } n > 0.
\]

By now we recognize this expression as a convolution, and we know how to patch the formula so that it holds for all integers \( n \):

\[
C_n = \sum_k C_k C_{n-1-k} + [n = 0]. \quad (7.65)
\]

Step 1 is now complete. Step 2 tells us to multiply by \( z^n \) and sum:

\[
C(z) = \sum_n C_n z^n
\]

\[
= \sum_{k,n} C_k C_{n-1-k} z^n + \sum_{n=0} z^n
\]

\[
= \sum_k C_k z^k \sum_n C_{n-1-k} z^{n-k} + 1
\]

\[
= C(z) \cdot z C(z) + 1.
\]

Lo and behold, the convolution has become a product, in the generating-function world. Life is full of surprises.

The authors jest.