Raney’s lemma can be proved by a simple geometric argument. Let’s extend the sequence periodically to get an infinite sequence

\[
\langle x_1, x_2, \ldots, x_m, x_1, x_2, \ldots, x_m, x_1, x_2, \ldots \rangle;
\]

thus we let \( x_{n+k} = x_k \) for all \( k \geq 0 \). If we now plot the partial sums \( s_n = x_1 + \cdots + x_n \) as a function of \( n \), the graph of \( s_n \) has an “average slope” of \( \frac{1}{m} \), because \( s_{m+n} = s_n + 1 \). For example, the graph corresponding to our example sequence \((3, -5, 2, -2, 3, 0, 3, -5, 2, \ldots)\) begins as follows:

![Graph of the example sequence](image)

The entire graph can be contained between two lines of slope \( \frac{1}{m} \), as shown; we have \( m = 6 \) in the illustration. In general these bounding lines touch the graph just once in each cycle of \( m \) points, since lines of slope \( \frac{1}{m} \) hit points with integer coordinates only once per \( m \) units. The unique lower point of intersection is the only place in the cycle from which all partial sums will be positive, because every other point on the curve has an intersection point within \( m \) units to its right.

With Raney’s lemma we can easily enumerate the sequences \( \langle a_0, \ldots, a_{2n} \rangle \) of \(+1\)'s and \(-1\)'s whose partial sums are entirely positive and whose total sum is \(+1\). There are \( \binom{2n+1}{n} \) sequences with \( n \) occurrences of \(-1\) and \( n+1 \) occurrences of \(+1\), and Raney’s lemma tells us that exactly \( \frac{1}{2n+1} \) of these sequences have all partial sums positive. (List all \( N = \binom{2n+1}{n} \) of these sequences and all \( 2n+1 \) of their cyclic shifts, in an \( N \times (2n+1) \) array. Each row contains exactly one solution. Each solution appears exactly once in each column. So there are \( N/(2n+1) \) distinct solutions in the array, each appearing \((2n+1)\) times.) The total number of sequences with positive partial sums is

\[
\binom{2n+1}{n} \frac{1}{2n+1} = \binom{2n}{n} \frac{1}{n+1} = C_n
\]

**Example 5: A recurrence with m-fold convolution.**

We can generalize the problem just considered by looking at sequences \( \langle a_0, \ldots, a_{m} \rangle \) of \(+1\)'s and \((1 \quad m)'s whose partial sums are all positive and

(Attention, computer scientists: The partial sums in this problem represent the stack size as a function of time, when a product of \( n+1 \) factors is evaluated, because each “push” operation changes the size by \(+1\) and each multiplication changes it by \(-1\).)