whose total sum is $+1$. Such sequences can be called m-Raney sequences. If there are $k$ occurrences of $(1 - m)$ and $mn + 1$ $k$ occurrences of $+1$, we have

$$k(1 - m) + (mn + 1 - k) = 1,$$

hence $k = n$. There are $\binom{mn + 1}{n}$ sequences with $n$ occurrences of $(1 - m)$ and $mn + 1$ $n$ occurrences of $+1$, and Raney’s lemma tells us that the number of such sequences with all partial sums positive is exactly

$$\binom{mn + 1}{n} \cdot \binom{mn}{n} \cdot \frac{1}{(m - 1)n + 1}.$$  \hspace{1cm} (7.66)

So this is the number of m-Raney sequences. Let's call this a Fuss-Catalan number $C_n^{(m)}$, because the sequence $(\ldots)$ was first investigated by N.I. Fuss [109] in 1791 (many years before Catalan himself got into the act). The ordinary Catalan numbers are $C_n = C_n^{(2)}$.

Now that we know the answer, (7.66), let's play “Jeopardy” and figure out a question that leads to it. In the case $m = 2$ the question was: “What numbers $C_n$ satisfy the recurrence $C_n = \sum_k C_k C_{n-k} + [n = 0]$?” We will try to find a similar question (a similar recurrence) in the general case.

The trivial sequence $(+1)$ of length 1 is clearly an m-Raney sequence. If we put the number $(1 - m)$ at the right of any m sequences that are m-Raney, we get an m-Raney sequence; the partial sums stay positive as they increase to $+2$, then $+3$, $\ldots$, $+m$, and $+1$. Conversely, we can show that all m-Raney sequences $(a_0, a_1, a_2, \ldots, a_n)$ arise in this way, if $n > 0$: The last term $a_n$, must be $(1 - m)$. The partial sums $s_j = a_0 + a_1, a_2 + a_3, \ldots + a_{j-1}$ are positive for $1 \leq j \leq mn$, and $s_{mn} = m$ because $s_{mn} + a_{mn} = 1$. Let $k_1$ be the largest index $\leq mn$ such that $s_{k_1} = 1$; let $k_2$ be the largest such that $s_{k_2} = 2$; and so on. Thus $s_{k_j} = j$ and $s_{k_j} > j$ for $k_j < k \leq mn$ and $1 \leq j \leq m$. It follows that $k_m = mn$, and we can verify without difficulty that each of the subsequences $(a_0, a_1, a_2, \ldots, a_{k_1-1})$, $(a_{k_1}, a_{k_2-1}, \ldots, a_{k_2-1})$, $(a_{k_2}, a_{k_3-1}, \ldots, a_{k_3-1})$, and so on, is an m-Raney sequence. We must have $k_1 = mn_1 + 1$, $k_2 = k_1 = mn_1 + 1$, $k_3 = k_2 = mn_1 + 1$, $\ldots$, $k_m = k_m - 1 = mn_m + 1$, for some nonnegative integers $n_1, n_2, \ldots, n_m$.

Therefore $\binom{mn + 1}{n_m}$ is the answer to the following two interesting questions: “What are the numbers $C_n^{(m)}$ defined by the recurrence

$$C_n^{(m)} = \sum_{n_1+n_2+\cdots+n_m=n-1} C_{n_1}^{(m)} C_{n_2}^{(m)} \cdots C_{n_m}^{(m)} + [n = 0]$$  \hspace{1cm} (7.67)

for all integers $n$?”  “If $G(z)$ is a power series that satisfies

$$G(z) = z G(z)^m + 1,$$  \hspace{1cm} (7.68)

what is $[z^n] G(z)$?”