Notice that these are not easy questions. In the ordinary Catalan case (m = 2), we solved (7.68) for \( G(z) \) and its coefficients by using the quadratic formula and the binomial theorem; but when m = 3, none of the standard techniques gives any clue about how to solve the cubic equation \( G = zG^3 + 1 \). So it has turned out to be easier to answer this question before asking it.

Now, however, we know enough to ask even harder questions and deduce their answers. How about this one: “What is \([z^n] G(z)^l\) if \( l \) is a positive integer and if \( G(z) \) is the power series defined by (7.68)?” The argument we just gave can be used to show that \([z^n] G(z)^l\) is the number of sequences of length \( mn + 1 \) with the following three properties:

- Each element is either +1 or (1, m).
- The partial sums are all positive.
- The total sum is 1.

For we get all such sequences in a unique way by putting together \( l \) sequences that have the m-Raney property. The number of ways to do this is

\[
\sum_{n_1 + n_2 + \cdots + n_l = n} C_{n_1}^{[m]} C_{n_2}^{[m]} \cdots C_{n_l}^{[m]} = [z^n] G(z)^l.
\]

Raney proved a generalization of his lemma that tells us how to count such sequences: If \( \langle x_1, x_2, \ldots, x_m \rangle \) is any sequence of integers with \( x_j \leq 1 \) for all \( j \), and with \( x_1 + x_2 + \ldots + x_m = 1 > 0 \), then exactly 1 of the cyclic shifts

\[\langle x_1, x_2, \ldots, x_m \rangle, \langle x_2, \ldots, x_m, x_1 \rangle, \ldots, \langle x_m, x_1, \ldots, x_{m-1} \rangle\]

have all positive partial sums.

For example, we can check this statement on the sequence \((-2, 1, -1, 0, 1, 1, -1, 1, 1, 1)\). The cyclic shifts are

\[
\begin{align*}
(-2, 1, -1, 0, 1, 1, -1, 1, 1, 1) & \quad (1, -1, 1, 1, -2, 1, -1, 0, 1) \\
(1, -1, 0, 1, 1, -1, 1, 1, -2) & \quad (-1, 1, 1, -2, 1, -1, 0, 1, 1) \\
(-1, 0, 1, 1, -1, 1, 1, -2, 1) & \quad (1, 1, 1, -2, 1, -1, 0, 1, 1, -1) \checkmark \\
(0, 1, 1, -1, 1, 1, -2, 1, -1) & \quad (1, 1, -2, 1, -1, 0, 1, 1, -1, 1) \\
(1, 1, -1, 1, 1, -2, 1, -1, 0) \checkmark & \quad (1, -2, 1, -1, 0, 1, 1, -1, 1, 1)
\end{align*}
\]

and only the two examples marked \( \checkmark \) have all partial sums positive. This generalized lemma is proved in exercise 13.

A sequence of \(+1\)'s and \( (1, m)\)'s that has length \( mn + 1 \) and total sum \( l \) must have exactly \( n \) occurrences of \( (1, 1) \) \( q \). The generalized lemma tells us that \( l/(mn + 1) \) of these \( \binom{mn + 1}{n} \) sequences have all partial sums positive;