this is the egf of \( \langle g_1, g_2, \ldots \rangle \). Thus differentiation on egf's corresponds to the left-shift operation \((G(z) - g_0)/z\) on ordinary gf's. (We used this left-shift property of egf's when we studied hypergeometric series, (5.106).) Integration of an egf gives

\[
\int_0^z \sum_{n \geq 0} g_n \frac{t^n}{n!} \, dt = \sum_{n \geq 0} g_n \frac{z^{n+1}}{(n+1)!} = \sum_{n \geq 1} g_{n-1} \frac{z^n}{n!};
\]

(7.73)

this is a right shift, the egf of \( \langle 0, g_0, g_1, \ldots \rangle \).

The most interesting operation on egf's, as on ordinary gf's, is multiplication. If \( \tilde{F}(z) \) and \( \tilde{G}(z) \) are egf's for \( \langle f_n \rangle \) and \( \langle g_n \rangle \), then \( \tilde{F}(z) \tilde{G}(z) = \tilde{H}(z) \) is the egf for a sequence \( \langle h_n \rangle \) called the binomial convolution of \( \langle f_n \rangle \) and \( \langle g_n \rangle \):

\[
h_n = \sum_k \binom{n}{k} f_k g_{n-k}
\]

(7.74)

Binomial coefficients appear here because \( \binom{n}{k} = \eta!/k!(n-k)! \), hence

\[
\frac{h_n}{n!} = \sum_{k=0}^n f_k \frac{g_{n-k}}{k!(n-k)!};
\]

in other words, \( \langle h_n/n! \rangle \) is the ordinary convolution of \( \langle f_n/n! \rangle \) and \( \langle g_n/n! \rangle \).

Binomial convolutions occur frequently in applications. For example, we defined the Bernoulli numbers in (6.79) by the implicit recurrence

\[
B_i = |m=0|, \quad \text{for all } m \geq 0;
\]

this can be rewritten as a binomial convolution, if we substitute \( \eta \) for \( m+1 \) and add the term \( B_n \) to both sides:

\[
\sum_k \binom{n}{k} B_k = B_n + |n=1|, \quad \text{for all } n \geq 0.
\]

(7.75)

We can now relate this recurrence to power series (as promised in Chapter 6) by introducing the egf for Bernoulli numbers, \( \tilde{B}(z) = \sum_{n \geq 0} B_n z^n/n! \). The left-hand side of (7.75) is the binomial convolution of \( \langle B_n \rangle \) with the constant sequence \( \langle 1, 1, 1, \ldots \rangle \); hence the egf of the left-hand side is \( \tilde{B}(z)e^z \). The egf of the right-hand side is \( \sum_{n \geq 0} (B_n + |n=1|)z^n/n! = \tilde{B}(z) + z \). Therefore we must have \( \tilde{B}(z) = z/(e^z - 1) \); we have proved equation (6.81), which appears also in Table 337 as equation (7.44).