7.2 Review of Matrices

For both theoretical and computational reasons it is advisable to bring to bear some of the results of matrix theory on the initial value problem for a system of linear differential equations. For reference purposes this section and the next are devoted to a brief summary of the facts that will be needed later. More details can be found in any elementary book on linear algebra. We assume, however, that you are familiar with determinants and how to evaluate them.

We designate matrices by boldfaced capitals \( A, B, C, \ldots \), occasionally using boldfaced Greek capitals \( \Phi, \Psi, \ldots \). A matrix \( A \) consists of a rectangular array of numbers, or elements, arranged in \( m \) rows and \( n \) columns, that is,

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}.
\]

(1)

We speak of \( A \) as an \( m \times n \) matrix. Although later in the chapter we will often assume that the elements of certain matrices are real numbers, in this section we assume that the elements of matrices may be complex numbers. The element lying in the \( i \)th row and \( j \)th column is designated by \( a_{ij} \), the first subscript identifying its row and the second its column. Sometimes the notation \( (a_{ij}) \) is used to denote the matrix whose generic element is \( a_{ij} \).

Associated with each matrix \( A \) is the matrix \( A^T \), known as the transpose of \( A \), and obtained from \( A \) by interchanging the rows and columns of \( A \). Thus, if \( A = (a_{ij}) \), then \( A^T = (a_{ji}) \). Also, we will denote by \( \overline{a}_{ij} \) the complex conjugate of \( a_{ij} \), and by \( \overline{A} \) the matrix obtained from \( A \) by replacing each element \( a_{ij} \) by its conjugate \( \overline{a}_{ij} \). The matrix \( \overline{A} \) is called the conjugate of \( A \). It will also be necessary to consider the transpose of the conjugate matrix \( \overline{A}^T \). This matrix is called the adjoint of \( A \) and will be denoted by \( A^* \).

For example, let

\[
A = \begin{pmatrix} 3 & 2 - i \\ 4 + 3i & -5 + 2i \end{pmatrix}.
\]

Then

\[
A^T = \begin{pmatrix} 3 & 4 + 3i \\ 2 - i & -5 + 2i \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 3 & 2 + i \\ 4 - 3i & -5 - 2i \end{pmatrix}, \quad A^* = \begin{pmatrix} 3 & 4 - 3i \\ 2 + i & -5 - 2i \end{pmatrix}.
\]

The properties of matrices were first extensively explored in 1858 in a paper by the English algebraist Arthur Cayley (1821–1895), although the word matrix was introduced by his good friend James Sylvester (1814–1897) in 1850. Cayley did some of his best mathematical work while practicing law from 1849 to 1863; he then became professor of mathematics at Cambridge, a position he held for the rest of his life. After Cayley’s groundbreaking work, the development of matrix theory proceeded rapidly, with significant contributions by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.