Now let’s look again at a sum that has been popping up frequently in this book,

\[ S_r(n) = \sum_{0 \leq k < n} k^m. \]

This time we will try to analyze the problem with generating functions, in hopes that it will suddenly become simpler. We will consider \( n \) to be fixed and \( m \) variable; thus our goal is to understand the coefficients of the power series

\[ S(z) = S_0(n) + S_1(n) z + S_2(n) z^2 + \cdots = \sum_{m \geq 0} S_m(n) z^m. \]

We know that the generating function for \((1, k, k^2, \ldots)\) is

\[ \frac{1}{1 - kz} = \sum_{m \geq 0} k^m z^m, \]

hence

\[ S(z) = \sum_{m \geq 0} \sum_{0 \leq k < n} k^m z^m = \sum_{0 \leq k < n} \frac{1}{1 - kz} \]

by interchanging the order of summation. We can put this sum in closed form,

\[ S(z) = \frac{1}{z} \left( \frac{1}{z - 1} + \frac{1}{z - 1 - 1} + \cdots + \frac{1}{z - 1 - n + 1} \right) \]

\[ = \frac{1}{z} \left( \frac{1}{H_{z-1}} - H_{z-1-n} \right); \quad (7.76) \]

but we know nothing about expanding such a closed form in powers of \( z \).

Exponential generating functions come to the rescue. The egf of our sequence \((S_0(n), S_1(n), S_2(n), \ldots)\) is

\[ \hat{S}(z, n) = S_0(n) + S_1(n) \frac{z}{1!} + S_2(n) \frac{z^2}{2!} + \cdots = \sum_{m \geq 0} S_m(n) \frac{z^m}{m!}. \]

To get these coefficients \( S_m(n) \) we can use the egf for \((1, k, k^2, \ldots)\), namely

\[ e^{kz} = \sum_{m \geq 0} \frac{k^m}{m!}, \]

and we have

\[ \hat{S}(z, n) = \sum_{m \geq 0} \sum_{0 \leq k < n} \frac{k^m}{m!} = \sum_{0 \leq k < n} e^{kz}. \]