And the latter sum is a geometric progression, so there’s a closed form

\[ S(z,n) = \frac{e^{nz} - 1}{e^z - 1} \tag{7.77} \]

All we need to do is figure out the coefficients of this relatively simple function, and we’ll know \( S_m[n] \), because \( S(n) = m! \left[ z^m \right] \hat{S}(z,n) \).

Here’s where Bernoulli numbers come into the picture. We observed a moment ago that the egf for Bernoulli numbers is

\[ \hat{B}(z) = \sum_{k \geq 0} \frac{B_k z^k}{k!} = \frac{z}{e^z - 1}; \]

hence we can write

\[
\hat{S}(z) = \hat{B}(z)^n z^{-1} = \left( B_0 \frac{z^0}{0!} + B_1 \frac{z^1}{1!} + B_2 \frac{z^2}{2!} + \cdots \right) \left( n \frac{z^n}{n!} + n^2 \frac{z^{n-1}}{2!} + n^3 \frac{z^{n-2}}{3!} + \cdots \right)
\]

The sum \( S(n) \) is \( m! \) times the coefficient of \( z^m \) in this product. For example,

\[
S_0(n) = 0! \left( B_0 \frac{n}{1!0!} \right) = n;
\]
\[
S_1(n) = 1! \left( B_0 \frac{n^1}{1!0!} + B_1 \frac{n}{1!1!} \right) = \frac{1}{2} n^2 - \frac{1}{2} n;
\]
\[
S_2(n) = 2! \left( B_0 \frac{n^2}{2!0!} + B_1 \frac{n^1}{1!1!} + B_2 \frac{n}{3!2!} \right) = \frac{1}{3} n^3 - \frac{1}{2} n^2 + \frac{1}{6} n.
\]

We have therefore derived the formula \( \square_n = S_2(n) = \frac{1}{3} n(n - \frac{1}{2})(n - 1) \) for the umpteenth time, and this was the simplest derivation of all: In a few lines we have found the general behavior of \( S(n) \) for all \( m \).

The general formula can be written

\[ S_{m-1}(n) = \frac{1}{m!} \left( B_m(n) - B_m(0) \right), \tag{7.78} \]

where \( B_m(x) \) is the Bernoulli polynomial defined by

\[ B_m(x) = \sum_k \binom{m}{k} B_k x^{m-k}. \tag{7.79} \]

Here’s why: The Bernoulli polynomial is the binomial convolution of the sequence \( \langle B_0, B_1, B_2, \ldots \rangle \) with \( \langle 1, x, x^2, \ldots \rangle \); hence the exponential generating