function for \(B_0(x), B_1(x), B_2(x), \ldots \) is the product of their egf's,

\[
\hat{B}(z, x) = \sum_{m \geq 0} B_m(x) \frac{z^m}{m!} = \frac{z}{e^z - 1} \sum_{m \geq 0} x^m \frac{z^m}{m!} = \frac{ze^{xz}}{e^z - 1}.
\] (7.80)

Equation (7.78) follows because the egf for \( (0, S_0(n), 2S_1(n), \ldots ) \) is, by (7.77),

\[
z \frac{e^{nz} - 1}{e^z - 1} = \hat{B}(z, n) \hat{B}(z, 0)
\]

Let's turn now to another problem for which egf's are just the thing:
How many spanning trees are possible in the complete graph on \( n \) vertices \( \{1, 2, \ldots, n\} \)? Let's call this number \( t_n \). The complete graph has \( \frac{1}{2}n(n - 1) \) edges, one edge joining each pair of distinct vertices; so we're essentially looking for the total number of ways to connect up \( n \) given things by drawing \( n - 1 \) lines between them.

We have \( t_1 = t_2 = 1 \). Also \( t_3 = 3 \), because a complete graph on three vertices is a fan of order 2; we know that \( f_2 = 3 \). And there are sixteen spanning trees when \( n = 4 \):

\[
\begin{align*}
\text{Fan} & \quad \text{Fan} & \quad \text{Fan} & \quad \text{Fan} \\
\text{Tree 1} & \quad \text{Tree 2} & \quad \text{Tree 3} & \quad \text{Tree 4}
\end{align*}
\]

Hence \( t_4 = 16 \).

Our experience with the analogous problem for fans suggests that the best way to tackle this problem is to single out one vertex, and to look at the blocks or components that the spanning tree joins together when we ignore all edges that touch the special vertex. If the non-special vertices form \( m \) components of sizes \( k_1, k_2, \ldots, k_m \), then we can connect them to the special vertex in \( k_1 k_2 \cdots k_m \) ways. For example, in the case \( n = 4 \), we can consider the lower left vertex to be special. The top row of (7.81) shows \( 3t_3 \) cases where the other three vertices are joined among themselves in \( t_3 \) ways and then connected to the lower left in 3 ways. The bottom row shows \( 2 \times t_2 t_1 x \binom{3}{2} \) solutions where the other three vertices are divided into components of sizes 2 and 1 in \( \binom{3}{2} \) ways; there's also the case \( \text{Tree 5} \) where the other three vertices are completely unconnected among themselves.

This line of reasoning leads to the recurrence

\[
t_n = \sum_{m > 0} \frac{1}{m!} \sum_{k_1 + k_2 + \cdots + k_m = n - 1} \binom{n - 1}{k_1, k_2, \ldots, k_m} k_1 k_2 \cdots k_m t_{k_1} t_{k_2} \cdots t_{k_m}
\]