Chapter 9. Inference in First-Order Logic

The proof answers the question "Did Curiosity kill the cat?" but often we want to pose more general questions, such as "Who killed the cat?" Resolution can do this, but it takes a little more work to obtain the answer. The goal is \( \neg \text{Kills}(w, \text{Tuna}) \) which, when negated, becomes \( \neg \text{Kills}(w, \text{Tuna}) \) in CNF. Repeating the proof in Figure 9.12 with the new negated goal, we obtain a similar proof tree, but with the substitution \( \{w/\text{Curiosity}\} \) in one of the steps. So, in this case, finding out who killed the cat is just a matter of keeping track of the bindings for the query variables in the proof.

Unfortunately, resolution can produce nonconstructive proofs for existential goals. For example, \( \neg \text{Kills}(w, \text{Tuna}) \) resolves with \( \text{Kills}(\text{Jack}, \text{Tuna}) \lor \text{Kills}(\text{Curiosity}, \text{Tuna}) \) to give \( \text{Kills}(\text{Jack}, \text{Tuna}) \), which resolves again with \( \neg \text{Kills}(w, \text{Tuna}) \) to yield the empty clause. Notice that \( w \) has two different bindings in this proof; resolution is telling us that yes, someone killed Tuna—either Jack or Curiosity. This is no great surprise! One solution is to restrict the allowed resolution steps so that the query variables can be bound only once in a given proof; then we need to be able to backtrack over the possible bindings. Another solution is to add a special answer literal to the negated goal, which becomes \( \neg \text{Kills}(w, \text{Tuna}) \lor \text{Answer}(w) \). Now, the resolution process generates an answer whenever a clause is generated containing just a single answer literal. For the proof in Figure 9.12, this is \( \text{Answer}(\text{Curiosity}) \). The nonconstructive proof would generate the clause \( \text{Answer}(\text{Curiosity}) \lor \text{Answer}(\text{Jack}) \), which does not constitute an answer.

9.5.4 Completeness of resolution

This section gives a completeness proof of resolution. It can be safely skipped by those who are willing to take it on faith.

We show that resolution is refutation complete, which means that if a set of sentences is unsatisfiable, then resolution will always be able to derive a contradiction. Resolution cannot be used to generate all logical consequences of a set of sentences, but it can be used to establish that a given sentence is entailed by the set of sentences. Hence, it can be used to find all answers to a given question, \( Q(x) \), by proving that \( \text{KB} \land \neg Q(x) \) is unsatisfiable.

We take it as given that any sentence in first-order logic (without equality) can be rewritten as a set of clauses in CNF. This can be proved by induction on the form of the sentence, using atomic sentences as the base case (Davis and Putnam, 1960). Our goal therefore is to prove the following: if \( S \) is an unsatisfiable set of clauses, then the application of a finite number of resolution steps to \( S \) will yield a contradiction.

Our proof sketch follows Robinson's original proof with some simplifications from Genesereth and Nilsson (1987). The basic structure of the proof (Figure 9.13) is as follows:

1. First, we observe that if \( S' \) is unsatisfiable, then there exists a particular set of ground instances of the clauses of \( S' \) such that this set is also unsatisfiable (Herbrand's theorem).
2. We then appeal to the ground resolution theorem given in Chapter 7, which states that propositional resolution is complete for ground sentences.
3. We then use a lifting lemma to show that, for any propositional resolution proof using the set of ground sentences, there is a corresponding first-order resolution proof using the first-order sentences from which the ground sentences were obtained.
Section 9.5. Resolution

Any set of sentences $S$ is representable in clausal form. Assume $S$ is unsatisfiable, and in clausal form.

Some set $S'$ of ground instances is unsatisfiable.

Resolution can find a contradiction in $S'$.

There is a resolution proof for the contradiction in $S'$.

Figure 9.13 Structure of a completeness proof for resolution.

To carry out the first step, we need three new concepts:

- **Herbrand universe**: If $S$ is a set of clauses, then $\Pi_S$, the Herbrand universe of $S$, is the set of all ground terms constructable from the following:
  
  a. The function symbols in $S$, if any.
  b. The constant symbols in $S$, if any; if none, then the constant symbol $\text{true}$.

  For example, if $S$ contains just the clause $\neg P(x, F(A, A)) \lor Q(A, A) \lor R(B, B)$, then $\Pi_S$ is the following infinite set of ground terms:

  $\{ (A, B, F(A, A), F(A, B), F(B, A), F(B, B), F(A, F(A, A)), \ldots ) \}$.

- **Saturation**: If $S$ is a set of clauses and $P$ is a set of ground terms, then $P(S)$, the saturation of $S$ with respect to $P$, is the set of all ground clauses obtained by applying all possible consistent substitutions of ground terms in $P$ with variables in $S$.

- **Herbrand base**: The saturation of a set $S$ of clauses with respect to its Herbrand universe is called the Herbrand base of $S$, written as $H_S(S)$. For example, if $S$ contains solely the clause just given, then $H_S(S)$ is the infinite set of clauses

  $\{ P(A, F(A, A)) \lor \neg Q(A, A) \lor R(A, B),
  F(B, A) \lor \neg Q(B, A) \lor R(B, B),
  \neg P(F(A, A), F(F(A, A), A)) \lor \neg Q(F(A, A), A) \lor R(F(A, A), B),
  \neg P(F(A, B), F(F(A, B), A)) \lor \neg Q(F(A, B), A) \lor R(F(A, B), B), \ldots \}$.

  These definitions allow us to state a form of Herbrand’s theorem (Herbrand, 1930):

  If a set $S$ of clauses is unsatisfiable, then there exists a finite subset of $H_S(S)$ that is also unsatisfiable.

Let $S'$ be this finite subset of ground sentences. Now, we can appeal to the resolution theorem (page 255) to show that the resolution closure $RC(S')$ contains the empty clause. That is, running propositional resolution to completion on $S'$ will derive a contradiction.

Now that we have established that there is always a resolution proof involving some finite subset of the Herbrand base of $S$, the next step is to show that there is a resolution