7.7 DIRICHLET GENERATING FUNCTIONS

There are many other possible ways to generate a sequence from a series; any system of “kernel” functions $K_n(z)$ such that

\[ \sum_n g_n K_n(z) = 0 \implies g_n = 0 \text{ for all } n \]

can be used, at least in principle. Ordinary generating functions use $K_n(z) = z^n$, and exponential generating functions use $K_n[z] = z^n/n!$; we could also try falling factorial powers $z^n$, or binomial coefficients $z^n/n! = \binom{z}{n}$.

The most important alternative to gf’s and egf’s uses the kernel functions $1/n^2$; it is intended for sequences $(g_1, g_2, \ldots)$ that begin with $n = 1$ instead of $n = 0$:

\[ \tilde{G}(z) = \sum_{n \geq 1} \frac{g_n}{n^2}. \quad (7.85) \]

This is called a Dirichlet generating function (dgf), because the German mathematician Gustav Lejeune Dirichlet (1805-1859) made much of it.

For example, the dgf of the constant sequence $(1, 1, 1, \ldots)$ is

\[ \sum_{n \geq 1} \frac{1}{n^2} = \zeta(z). \quad (7.86) \]

This is Riemann’s zeta function, which we have also called the generalized harmonic number $H(z)$ when $z > 1$.

The product of Dirichlet generating functions corresponds to a special kind of convolution:

\[ \tilde{F}(z) \tilde{G}(z) = \sum_{l,m \geq 1} \frac{f_l}{l^2} \cdot \frac{g_m}{m^2} = \sum_{n \geq 1} \frac{1}{n^2} \sum_{l,m \geq 1} f_l g_m \cdot [l \cdot m = n]. \]

Thus $\tilde{F}(z) \tilde{G}(z) = \tilde{H}(z)$ is the dgf of the sequence

\[ h_n = \sum_{d \mid n} \mu(n/d). \quad (7.87) \]

For example, we know from (4.55) that $\sum_{d \mid n} \mu(d) = [n = 1]$; this is the Dirichlet convolution of the Mobius sequence $(\mu(1), \mu(2), \mu(3), \ldots)$ with $(1, 1, 1, \ldots)$, hence

\[ \tilde{M}(z) \zeta(z) = \sum_{n \geq 1} \frac{[n = 1]}{n^2} = 1. \quad (7.88) \]

In other words, the dgf of $(\mu(1), \mu(2), \mu(3), \ldots)$ is $\zeta(z)^{-1}$. 