Dirichlet generating functions are particularly valuable when the sequence \((g_1, g_2, \ldots)\) is a \textit{multiplicative function}, namely when
\[
g_{mn} = g_m g_n \quad \text{for } m \perp n.
\]
In such cases the \textit{values} of \(g_n\) for all \(n\) are determined by the values of \(g_n\) when \(n\) is a power of a prime, and we can factor the dgf into a product over primes:
\[
\tilde{G}(z) = \prod_{p \text{ prime}} \left( 1 + \frac{g_p}{p^z} + \frac{g_{p^2}}{p^{2z}} + \frac{g_{p^3}}{p^{3z}} + \cdots \right) \quad (7.8g)
\]
If, for instance, we set \(g_n = 1\) for all \(n\), we obtain a product representation of Riemann’s zeta function:
\[
\zeta(z) = \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-z}} \right) \quad (7.90)
\]
The Mobius function has \(\mu(p) = -1\) and \(\mu(p^k) = 0\) for \(k > 1\), hence its dgf is
\[
\tilde{M}(z) = \prod_{p \text{ prime}} \left( 1 - p^{-z} \right) \quad (7.91)
\]
this agrees, of course, with (7.88) and (7.90). Euler’s \(\varphi\) function has \(\varphi(p^k) = p^k - p^{k-1}\), hence its dgf has the factored form
\[
\tilde{\Phi}(z) = \prod_{p \text{ prime}} \left( 1 + \frac{p - 1}{p^z - p} \right) = \prod_{p \text{ prime}} \frac{1 - p^{-z}}{1 - p^{-1}} \quad (7.92)
\]
We conclude that \(\tilde{\Phi}(z) = \zeta(z-1)/\zeta(z)\).

\section*{Exercises}
\subsection*{Warmups}
1. An eccentric collector of 2 x \(n\) domino tilings pays $4 for each vertical domino and $1 for each horizontal domino. How many tilings are worth exactly $m$ by this criterion? For example, when \(m = 6\) there are three solutions: $\blacksquare$, $\blacksquare\blacksquare$, and $\blacksquare\blacksquare\blacksquare$.
2. Give the generating function and the exponential generating function for the sequence \((2, 5, 13, 35, \ldots) = (2^n + 3^n)\) in closed form.
3. What is \(\sum_{n \geq 0} H_n/10^n\)?
4. The general expansion theorem for rational functions \(P(z)/Q(z)\) is not completely general, because it restricts the degree of \(P\) to be less than the degree of \(Q\). What happens if \(P\) has a larger degree than this?