Find a generating function $S(z)$ such that

$$[z^n] S(z) = \sum_k \binom{r}{k} (k,1) .$$

**Basics**

Show that the recurrence (7.32) can be solved by the repertoire method, without using generating functions.

Solve the recurrence

$$g_0 = 1 ;$$
$$g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0 , \quad \text{for } n > 0 .$$

What is $[z^n] (\ln(1 - z))^2/(1 - z)^{m+1}$?

Use the result of the previous exercise to evaluate $\sum_{k=0}^{n} H_k H_{n-k}$.

Set $\tau = s = -1/2$ in identity (7.61) and then remove all occurrences of $1/2$ by using tricks like (5.36). What amazing identity do you deduce?

This problem, whose three parts are independent, gives practice in the manipulation of generating functions. We assume that $A(z) = \sum a_n z^n$, $B(z) = \sum b_n z^n$, $C(z) = \sum c_n z^n$, and that the coefficients are zero for negative $n$.

- If $c_n = \sum_{i+2k \leq n} a_i b_k$, express $C$ in terms of $A$ and $B$.
- If $b_n = \sum_{k=0}^{n} \frac{2^n a_k}{(n-k)!}$, express $A$ in terms of $B$.
- If $\tau$ is a real number and if $a_n = \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k}$, express $A$ in terms of $B$; then use your formula to find coefficients $f_k(\tau)$ such that $b_n = \sum_{k=0}^{n} f_k(\tau) a_{n-k}$.

How many ways are there to put the numbers $\{1,2, \ldots, 2n\}$ into a $2 \times n$ array so that rows and columns are in increasing order from left to right and from top to bottom? For example, one solution when $n = 5$ is

$$\begin{pmatrix} 1 & 2 & 4 & 5 & 8 \\ 3 & 6 & 7 & 9 & 10 \end{pmatrix} .$$

Prove Raney’s generalized lemma, which is stated just before (7.6g).

Solve the recurrence

$$g_0 = 0 , \quad g_1 = 1 ,$$
$$g_n = -2ng_{n-1} + \sum_k \binom{n}{k} g_k g_{n-k} , \quad \text{for } n > 1 ,$$

by using an exponential generating function.