Chapter 7. Systems of First Order Linear Equations

\[
\begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{3}{2} & \frac{1}{2} & 0 \\
4 & -2 & 1
\end{pmatrix},
\begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{3}{2} & \frac{1}{2} & 0 \\
-\frac{4}{5} & \frac{7}{5} & -\frac{1}{5}
\end{pmatrix},
\begin{pmatrix}
\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\
\frac{1}{10} & -\frac{1}{10} & \frac{1}{10} \\
-\frac{4}{5} & \frac{7}{5} & -\frac{1}{5}
\end{pmatrix}.
\]

The last of these matrices is \( A^{-1} \), a result that can be verified by direct multiplication with the original matrix \( A \).

This example is made slightly simpler by the fact that the original matrix \( A \) had a 1 in the upper left corner \( (a_{11} = 1) \). If this is not the case, then the first step is to produce a 1 there by multiplying the first row by \( 1/a_{11} \), as long as \( a_{11} \neq 0 \). If \( a_{11} = 0 \), then the first row must be interchanged with some other row to bring a nonzero element into the upper left position before proceeding.

**Matrix Functions.** We sometimes need to consider vectors or matrices whose elements are functions of a real variable \( t \). We write

\[
x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix},
\]

respectively.

The matrix \( A(t) \) is said to be continuous at \( t = t_0 \) or on an interval \( \alpha < t < \beta \) if each element of \( A \) is a continuous function at the given point or on the given interval. Similarly, \( A(t) \) is said to be differentiable if each of its elements is differentiable, and its derivative \( dA/dt \) is defined by

\[
dA/dt = \begin{pmatrix} \partial a_{ij} / \partial t \\ \vdots \\ \partial a_{nj} / \partial t \end{pmatrix};
\]

that is, each element of \( dA/dt \) is the derivative of the corresponding element of \( A \). In the same way the integral of a matrix function is defined as

\[
\int_a^b A(t) \, dt = \left( \int_a^b a_{ij}(t) \, dt \right).
\]

For example, if

\[
A(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix},
\]

then

\[
A'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix}, \quad \int_0^\pi A(t) \, dt = \left( \frac{2}{\pi} \pi^2/2 \right).
\]

Many of the rules of elementary calculus extend easily to matrix functions; in particular,

\[
\frac{d}{dt} (CA) = C \frac{dA}{dt}, \quad \text{where } C \text{ is a constant matrix;}
\]

\[
\frac{d}{dt} (A + B) = \frac{dA}{dt} + \frac{dB}{dt};
\]

\[
\frac{d}{dt} (AB) = A \frac{dB}{dt} + A \frac{dA}{dt}.
\]