A power series $G(z)$ is called differentiably finite if there exist finitely many polynomials $P_0(z), \ldots, P_m(z)$, not all zero, such that

$$P_0(z)G(z) + P_1(z)G'(z) + \cdots + P_m(z)G^{(m)}(z) = 0.$$ 

A sequence of numbers $(g_0, g_1, g_2, \ldots)$ is called polynomially recursive if there exist finitely many polynomials $p_0(z), \ldots, p_m(z)$, not all zero, such that

$$p_0(n)g_n + p_1(n)g_{n+1} + \cdots + p_m(n)g_{n+m} = 0$$

for all integers $n \geq 0$. Prove that a generating function is differentiably finite if and only if its sequence of coefficients is polynomially recursive.

**Homework exercises**

1. A robber holds up a bank and demands $500 in tens and twenties. He also demands to know the number of ways in which the cashier can give him the money. Find a generating function $G(z)$ for which this number will he settle for is $[z^{500}]G(z)$, and a more compact generating function $\hat{G}(z)$ for which this number is $[z^{50}]\hat{G}(z)$. Determine the required number of ways by (a) using partial fractions; (b) using a method like (7.39).

2. Let $P$ be the sum of all ways to “triangulate” polygons:

$$P = \_ + \triangle + \square + \square + \square + \square + \square + \square + \cdots.$$ 

(The first term represents a degenerate polygon with only two vertices; every other term shows a polygon that has been divided into triangles. For example, a pentagon can be triangulated in five ways.) Define a “multiplication” operation $A\Delta B$ on triangulated polygons $A$ and $B$ so that the equation

$$P = \_ + P A P$$

is valid. Then replace each triangle by ‘z’; what does this tell you about the number of ways to decompose an $n$-gon into triangles?

3. In how many ways can a 2 x 2 x n pillar be built out of 2 x 1 x 1 bricks?

4. How many spanning trees are in an $n$-wheel (a graph with $n$ “outer” vertices in a cycle, each connected to an $(n+1)$st “hub” vertex), when $n \geq 3$?