25 Let $m \geq 2$ be an integer. What is a closed form for the generating function of the sequence $(n \mod m)$, as a function of $z$ and $m$? Use this generating function to express `$n \mod m$' in terms of the complex number $\omega = e^{\pi i / m}$. (For example, when $m = 2$ we have $\omega = -1$ and $n \mod 2 = \frac{1}{2} - \frac{1}{2}(-1)^n$.)

26 The second-order Fibonacci numbers $\{\mathcal{F}_n\}$ are defined by the recurrence

$$\begin{align*}
\mathcal{F}_0 &= 0; \\
\mathcal{F}_1 &= 1; \\
\mathcal{F}_n &= \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + F_n, \quad \text{for } n > 1.
\end{align*}$$

Express $\mathcal{F}_n$ in terms of the usual Fibonacci numbers $F_n$ and $F_{n+1}$.

27 A $2 \times n$ domino tiling can also be regarded as a way to draw $n$ disjoint lines in a $2 \times n$ array of points:

```
  I I I I II
```

If we superimpose two such patterns, we get a set of cycles, since every point is touched by two lines. For example, if the lines above are combined with the lines

```
  I I I I II
```

the result is

```
  I I I I II
```

The same set of cycles is also obtained by combining

```
  I I I I II
```

with

```
  I I I I II
```

But we get a unique way to reconstruct the original patterns from the superimposed ones if we assign orientations to the vertical lines by using arrows that go alternately up/down/up/down/... in the first pattern and alternately down/up/down/up/... in the second. For example,

```
  I I I I II + I I I I II = I I I I II
```

The number of such oriented cycle patterns must therefore be $T_n^2 = F_{n+1}^2$, and we should be able to prove this via algebra. Let $Q_n$ be the number of oriented $2 \times n$ cycle patterns. Find a recurrence for $Q_n$, solve it with generating functions, and deduce algebraically that $Q_n = F_{n+1}^2$.

28 The coefficients of $A(z)$ in (7.39) satisfy $A_r + A_{r+10} + A_{r+20} + A_{r+30} = 100$ for $0 \leq r < 10$. Find a “simple” explanation for this.