37 Let $a_n$ be the number of ways to write the positive integer $n$ as a sum of powers of 2, disregarding order. For example, $a_4 = 4$, since $4 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. By convention we let $a_0 = 1$. Let $b_n = \sum_{k=0}^n a_k$ be the cumulative sum of the first $a$'s.

a. Make a table of the $a$'s and $b$'s up through $n = 10$. What amazing relation do you observe in your table? (Don't prove it yet.)

b. Express the generating function $A(z)$ as an infinite product.

c. Use the expression from part (b) to prove the result of part (a).

38 Find a closed form for the double generating function

$$M(w, z) = \sum_{m,n \geq 0} \min(m,n) w^m z^n$$

Generalize your answer to obtain, for fixed $m \geq 2$, a closed form for

$$M(z_1, \ldots, z_m) = \sum_{n_1, \ldots, n_m \geq 0} \min(n_1, \ldots, n_m) z_1^{n_1} \cdots z_m^{n_m}.$$ 

39 Given positive integers $m$ and $n$, find closed forms for

$$\sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} k_1 k_2 \ldots k_m$$

and

$$\sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq n} k_1 k_2 \ldots k_m.$$ 

(For example, when $m = 2$ and $n = 3$ the sums are $1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3$ and $1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 3 + 3 \cdot 3$.) Hint: What are the coefficients of $z^m$ in the generating functions $(1 + a_1 z) \ldots (1 + a_n z)$ and $1/(1 - a_1 z) \ldots (1 - a_n z)$?

40 Express $\sum_k \binom{n}{k} [k F_{k-1} - F_k] (n-k)!$ in closed form.

41 An up-down permutation of order $n$ is an arrangement $a_1 a_2 \ldots a_n$, of the integers $\{1, 2, \ldots, n\}$ that goes alternately up and down:

$$a_1 < a_2 > a_3 < a_4 > \cdots$$

For example, 35142 is an up-down permutation of order 5. If $A_n$ denotes the number of up-down permutations of order $n$, show that the exponential generating function of $(A_n)$ is $(1 + \sin z)/\cos z$.

42 A space probe has discovered that organic material on Mars has DNA composed of five symbols, denoted by (a, b, c, d, e), instead of the four components in earthing DNA. The four pairs cd, ce, ed, and ee never occur consecutively in a string of Martian DNA, but any string without forbidden pairs is possible. (Thus bbcd is forbidden but bbdea is OK.) How many Martian DNA strings of length $n$ are possible? (When $n = 2$ the answer is 21, because the left and right ends of a string are distinguishable.)