43 The Newtonian generating function of a sequence \( \langle g_n \rangle \) is defined to be
\[
\hat{G}(z) = \sum_n g_n \binom{z}{n}
\]
Find a convolution formula that defines the relation between sequences \( \langle f_n \rangle \), \( \langle g_n \rangle \), and \( \langle h_n \rangle \) whose Newtonian generating functions are related by the equation \( \hat{F}(z)\hat{G}(z) = \hat{H}(z) \). Try to make your formula as simple and symmetric as possible.

44 Let \( q_n \) be the number of possible outcomes when \( n \) numbers \( \{x_1, \ldots, x_n\} \) are compared with each other. For example, \( q_3 = 13 \) because the possibilities are
\[
\begin{align*}
x_1 < x_2 < x_3; \quad & x_1 < x_2 = x_3; \quad x_1 < x_3 < x_2; \quad x_1 = x_2 < x_3; \\
x_1 = x_2 = x_3; \quad & x_1 = x_3 < x_2; \quad x_2 < x_1 < x_3; \\
x_2 < x_1 = x_3; \quad & x_2 < x_3 < x_1; \quad x_2 = x_3 < x_1; \\
x_3 < x_1 < x_2; \quad & x_3 < x_1 = x_2; \quad x_3 < x_2 < x_1.
\end{align*}
\]
Find a closed form for the egf \( \hat{Q}(z) = \sum_n q_n z^n/n! \). Also find sequences \( \langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle \) such that
\[
q_n = \sum_{k=0}^{n} k^n a_k = \sum_k \binom{n}{k} b_k = \sum_k \binom{n}{k} c_k, \quad \text{for all } n > 0.
\]

45 Evaluate \( \sum_{m,n>0} (m
\perp n)/m^2n^2 \).

46 Evaluate
\[
\sum_{0 \leq k \leq n/2} \binom{n-2k}{k} \left(\frac{-4}{27}\right)^k
\]
in closed form. Hint: \( z^3 + z^2 + \frac{4}{27} = \left(z + \frac{1}{3}\right)(z - \frac{2}{3})^2 \).

47 Show that the numbers \( \mathcal{U}_n \) and \( \mathcal{V}_n \) of 3 \( \times \) \( n \) domino tilings, as given in (7.34), are closely related to the fractions in the Stern-Brocot tree that converge to \( \sqrt{3} \).

48 A certain sequence \( \langle g_n \rangle \) satisfies the recurrence
\[
a g_n + b g_{n+1} + c g_{n+2} + d = 0, \quad \text{integer } n \geq 0,
\]
for some integers \( a, b, c, d \) with \( \gcd(a, b, c, d) = 1 \). It also has the closed form
\[
g_n = [\alpha(1 + \sqrt{2})]^n, \quad \text{integer } n \geq 0,
\]
for some real number \( \alpha \) between 0 and 1. Find \( a, b, c, d, \) and \( \alpha \).